Bond Valuation Under Discrete-Time Regime-Switching Term-Structure Models

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Outline of the presentation:

- Literature review
- Motivation
- Markovian-regime switching for interest rate
- Change of measure
- Bond valuation formula
- Summary

§1. Literature review

- **Single-factor models** - early models for describing the stochastic behavior of interest rates (e.g. Merton (1973), Vasicek (1977), Cox, Ingersoll and Ross (1985)) - closed-form pricing formula for zero-coupon bonds are available.

- Imply that the prices of bonds with different maturities are perfectly correlated - not consistent with the empirical behavior of yield curves

- Volatility structure of the short rate is more complex than those arising from the single-factor models (e.g. Chapman and Pearson (2001))

- **Multi-factor models** - the evolution of the term structure of interest rates over time depends on several (macro-economic) factors (e.g. Brennan and Schwartz (1979), Longstaff and Schwartz (1992))

- Chen (1996) - three-factor model with factors including the short rate, the short-term mean rate and the short rate volatility

- Duffie and Kan (1996) - a general multi-factor affine term structure model, which nests many of the single-factor and multi-factor models in the existing literature

- Many existing stochastic term structure models were primarily developed for pricing, hedging and risk management of some short or medium-term interest-rate products

- For modelling the long-term behavior of interest rates and managing the risk of long-term interest-rate related products including insurance products, it is important to model the long-term behavior of interest rates

- For a long time period, there may be structural changes in the term structure of interest rates, which can be attributed to the structural changes of macro-economic conditions

- **Regime-switching models** - capture these structural changes of macro-economic conditions, economic fundamentals and monetary policies (e.g. Hamilton (1989), Evans and Lewis (1991), Ang and Bekaert 2002a, b)

- Ang and Bekaert (2002a) study the performance of regime-switching models in fitting interest rate data from the United States

- Switching of regimes in interest rates matches well with business cycles in the United States and that regime-switching models have better out-of-sample forecasts than models without switching regimes

- It is desirable to have a term structure model, which is flexible and general enough to describe various empirical features of interest rate data and is tractable for valuation

- Duffie and Kan (1996) - if the price of a zero coupon bond has an exponential-affine form, then the underlying short rate dynamics must be of either linear/Gaussian form or square-root/affine form

- Elliott and van der Hoek (2001) - if the continuous-time models for short rates have either a linear/Gaussian form or a square-root/affine form, the bond price has an exponential-affine form.

- **Discrete-time term structure models** - seem more easy to implement and estimate than their continuous-time counterparts (e.g. Evans 1998, Dai and Singleton 2000, and Ang and Bekaert 2002c)

- Dai et al. (2007) - empirically examined a discrete time, regime switching model where the short rate follows a three-factor Gaussian model with state-dependent market prices of risk

- Ang et al. (2007) - employed a three-factor representation for yield curves by incorporating regime changes for the inflation factor and the term structure factor, and assuming a regime-invariant price of risk for the factors

§2. Motivation

- There are two sources of risk one should take into account for pricing interest derivatives

- By analogy with continuous time, most of discrete-time research studies rely on an exponential affine form for bond prices

§3. Markovian-regime switching for interest rate

- Consider a discrete-time economy with time index set $\mathcal{T} = \{k|k = 0, 1, 2, \cdots, T\}$

- Describe the evolution of the states of the economy by a discrete-time, N-state, Markov chain $\mathbf{X} := \{\mathbf{X}_k\}_{k\in\mathcal{T}}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\}$ a finite set of unit vectors with $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)' \in \Re^N$

- $\mathbf{A} := \{a_{ij}\}_{i,j=1,2,\cdots,N}$, transition probability matrix under P

$$a_{ij} = P(\mathbf{X}_{k+1} = \mathbf{e}_j|\mathbf{X}_k = \mathbf{e}_i)$$

- Let $\mathcal{F}^{\mathbf{X}} := \{\mathcal{F}_k^{\mathbf{X}}\}_{k \in \mathcal{T}}$ denote the P-augmented filtration generated by the history of the chain $\mathbf{X}$

- The semimartingale representation of the chain $\mathbf{X}$ (Elliott et al., 1994):

$$\mathbf{X}_{k+1} = \mathbf{A}\mathbf{X}_k + \mathbf{M}_{k+1}$$

  Here $\mathbf{M} := \{\mathbf{M}_k\}_{k=1,2,\cdots,T}$ is an $\Re^N$-valued martingale increment process so that

$$E[\mathbf{M}_{k+1} | \mathcal{F}_k^{\mathbf{X}}] = 0 \ .$$

- Define

$$\begin{aligned} \boldsymbol{\alpha} &:= (\alpha_1, \alpha_2, \cdots, \alpha_N)' \in \Re^N \\ \boldsymbol{\beta} &:= (\beta_1, \beta_2, \cdots, \beta_N)' \in \Re^N \\ \boldsymbol{\sigma} &:= (\beta_1, \beta_2, \cdots, \beta_N)' \in \Re^N \end{aligned}$$

- $\varepsilon := \{\varepsilon_k\}_{k=1,2,\cdots,T}$ a sequence of iid random variable with $\varepsilon_k \sim N(0,1)$

- ε and $\mathbf{X}$ are stochastically independent

- Let $r := \{r_k\}_{k\in\mathcal{T}}$ denote the process of short term interest rate

- $\mathcal{F}^r := \{\mathcal{F}^r_k\}_{k\in\mathcal{T}}$ for the P-augmented filtration generated by the interest rate process r

- $\mathcal{G}_k = \mathcal{F}^r_k \vee \mathcal{F}^{\mathbf{X}}_k$ - current and historical information about both the short term interest rates and the economic regimes - observed by economic agents

- Let $\boldsymbol{\theta}_k := (\theta_{1k}, \theta_{2k}, \cdots, \theta_{Nk})' \in \Re^N$, for $k = 1, 2, \cdots, T$

- θ_{ik} is $\mathcal{G}_{k-1}$-measurable - the price of risk at time k when the state of the economy is in the i^{th} regime at time k

- $\theta_k := \langle \boldsymbol{\theta}_k, \mathbf{X}_k \rangle$

- For any $k = 0, 1, \cdots, T-1$, under P:

  $$r_{k+1} = \langle \boldsymbol{\alpha}, \mathbf{X}_k \rangle + \langle \boldsymbol{\theta}_k, \mathbf{X}_k \rangle \langle \boldsymbol{\sigma}, \mathbf{X}_k \rangle^2 + \langle \boldsymbol{\beta}, \mathbf{X}_k \rangle r_k + \langle \boldsymbol{\sigma}, \mathbf{X}_k \rangle \varepsilon_{k+1}$$

- In practice one may assume $\boldsymbol{\theta}_k$ does not depend on time - $\boldsymbol{\theta}_k = \boldsymbol{\theta}$

§4. Change of measure

- Two sources of risk: Diffusion risk and the regime-switching risk

- Incomplete market - more than one equivalent martingale measures

- Key question: How to select an equivalent martingale measure so that the regime-switching risk and the diffusion risk are priced?

- Generate a family of probability measures by a product of two density processes, one for r and one for the Markov chain X

- Introduce a density process for r

- Let $\Lambda^{\theta} = \{\Lambda_k^{\theta}\}_{k \in \mathcal{T}}$ a G-adapted processes:

$$\begin{aligned} \Lambda_0^{\theta} &= 1 \\ \Lambda_k^{\theta} &:= \prod_{t=1}^{k} \lambda_t^{\theta} \\ &= \exp\left(-\frac{1}{2} \sum_{t=1}^{k} \theta_{t-1}^2 \sigma_{t-1}^2 - \sum_{t=1}^{k} \theta_{t-1} \sigma_{t-1} \varepsilon_t \right) \end{aligned}$$

- Λ^{θ} is a $(\mathcal{G}, P)$-martingale

- Λ^{θ} is a discrete-time version of the Girsanov's change of measures in a continuous-time setting

- Introduce a density process for the Markov chain X

- For each $t \in \mathcal{T}$, let $\mathbf{\Delta}_t := (\boldsymbol{\delta}_{ijt})_{1 \leq i,j \leq N}$, be an $(N \times N)$-matrix of $\mathcal{G}_t$-measurable components

- Write $\boldsymbol{\delta}_t = \mathbf{\Delta}_t \mathbf{X}_t \in \Re^N$

- Define $\lambda_{ij,t-1} := \exp(\left\langle \mathbf{\Delta}_{t-1}\mathbf{e}_j, \mathbf{e}_i \right\rangle) = \exp(\delta_{ij,t-1})$, $\quad i,j = 1, 2, \cdots, N$

- Write $\mathbf{L}_{t-1} := (\lambda_{ij,t-1})_{i,j=1,2,\cdots,N}$

- Let $\Lambda^{\boldsymbol{\delta}} := \{\Lambda_k^{\boldsymbol{\delta}}\}_{k\in\mathcal{T}}$ a G-adapted process:

$$\begin{aligned}\Lambda_0^{\boldsymbol{\delta}} &= 1\\ \Lambda_k^{\boldsymbol{\delta}} &:= \prod_{t=1}^{k} \lambda_t^{\boldsymbol{\delta}} = \prod_{t=1}^{k}\left(1+\frac{\mathbf{X}_{t-1}'\mathbf{L}_{t-1}'\mathbf{M}_t}{\mathbf{X}_{t-1}'(\mathbf{L}_{t-1}'\mathbf{A})\mathbf{X}_{t-1}}\right)\end{aligned}$$

- $\Lambda^{\boldsymbol{\delta}}$ is a $(\mathcal{G}, P)$-martingale

- define the $(\mathcal{G}, P)$-adapted process $\Lambda^{\theta,\boldsymbol{\delta}} := \{\Lambda_k^{\theta,\boldsymbol{\delta}}\}_{k\in\mathcal{T}}$ by the product of the density processes Λ^{θ} and $\Lambda^{\boldsymbol{\delta}}$ as:

$$\Lambda_k^{\theta,\boldsymbol{\delta}} = \Lambda_k^{\theta}\cdot\Lambda_k^{\boldsymbol{\delta}}$$

- $\Lambda^{\theta,\delta}$ is a $(\mathcal{G}, P)$-martingale

- We can define a new probability measure $Q \sim P$ on $\mathcal{G}_k$, for each $k \in \mathcal{T}$, by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{G}_k} := \Lambda_k^{\theta,\boldsymbol{\delta}}$$

- This construction of Q extends one proposed in (Elliott et al. 2005) for option valuation when the return process is governed by a regime-switching Gaussian process

- We accommodate our method to price two sources of risk

- Price of risk corresponding to the short term interest rate r_k is embedded in the parameter θ_k, while $\boldsymbol{\delta}_k$ represents the market price of the regime shift from $\mathbf{X}_{k-1}$ to $\mathbf{X}_k$

Proposition 1

(a) Under the risk-neutralized probability measure Q, the short term interest rate has the following dynamics:

$$r_{t+1} = \langle \boldsymbol{\alpha}, \mathbf{X}_t \rangle + \langle \boldsymbol{\beta}, \mathbf{X}_t \rangle \, r_t + \langle \boldsymbol{\sigma}, \mathbf{X}_t \rangle \, \varepsilon_{t+1}^{\dagger} \ , \qquad t = 0, 1, \cdots, T-1 \ ,$$

$\varepsilon_t^{\dagger}$'s are independent and identically distributed, Gaussian, random variables which are independent of $\mathbf{X}_t$.

(b) The elements of the transition matrix $\mathbf{C}_t = (c_{ijt})_{1 \le i,j \le N}$ of the Markov chain $\mathbf{X}$ under Q are given by:

$$c_{ijt} = \frac{a_{ij} \exp(\delta_{ijt})}{\sum\limits_{j=1}^{N} a_{ij} \exp(\delta_{ijt})}$$

§5. Bond valuation formula

- Let $B(k,T)$ the time-k price of a zero-coupon bond with maturity at time T

- Write $Z(k,n)$ for the discounted bond price at time k:

$$Z(k,T) := \text{exp}\left(- \sum_{t=1}^{k} r_t \right) B(k,T) \ .$$

- If Q is a new probability measure under which the discounted bond price process $Z(T) := \{Z(k,T)\}_{k \in \mathcal{T}}$ is a martingale with respect to $\mathcal{G}$ then:

$$B(k,T) = E^Q \left[\text{exp} \left(- \sum_{t=k+1}^{T} r_t \right) | \mathcal{G}_k \right]$$

- **Example - Continuous time**

- Consider a continuous-time financial market with a finite time horizon $\mathcal{T} := [0, T]$

- Let $X := X(t)_{t \in \mathcal{T}}$ be a continuous-time, finite-state observable Markov chain on $(\Omega, \mathcal{F}, \mathcal{P})$

- $X(t) = X(0) + \int_0^t QX(s)ds + M(t)$

- Q is the rate matrix and $M(t)_{t \in \mathcal{T}}$ is a P-martingale wrt the filtration generated by X

- $dr(t) = a(\alpha(X(t)) - r(t))dt + \sigma(X(t))dW(t)$

- Denote by $P(t, T, X, r)$ the price of a zero-coupon bond at time t with maturity T given the values at t are X and r

- $P(t, T, X, r) = e^{A(t,T,X) - B(t,T)r}$

- $B(t, T) = \frac{1}{a}(1 - e^{-a(T-t)})$

- $A(t, T, X)$ satisfies the following regime switching O.D.E:

  $$\frac{dA}{dt} - \alpha(t)aB(t, T) + \frac{1}{2}\sigma^2(t)B^2(t, T) + e^{-A}\langle \mathcal{K}, QX(t)\rangle = 0$$

  where $A(T, T, X) = 0$ and $K(t, T, X) := e^{A(t,T,X)}$

- Traditional approach in discrete time

- Assume $\langle \boldsymbol{\beta}, \mathbf{X}_k \rangle = \beta$

- Conjecture the bond price is given by:

$$B(k,T) = e^{A_{1k}(X_k) + A_{2k} r_k}$$

- Find recursive relationships between the coefficients from the exponential-affine formula by imposing the martingale condition on the discounted bond price process $Z(T)$

- Dai, Q., Singleton, K.J., Yang, W. (2007), Ang, A., Bekaert, G., Wei, M. (2007)

- Following this approach for any $k = 1, 2, \cdots, T$ we find:

$$\begin{aligned} A_{2k-1} &= A_{2k}(\beta - 1) \\ A_{1k-1}(X_{k-1}) &= (A_{2k} - 1)\Big(\langle \alpha, \mathbf{X}_{k-1} \rangle + \frac{1}{2}(A_{2k} - 1) \langle \sigma, \mathbf{X}_{k-1} \rangle^2 \Big) \\ &+ \log \sum_{X_k} c_{X_{k-1} X_k k} e^{A_{1k}(X_k)} \end{aligned}$$

  with initial conditions: $A_{1T} = A_{2T} = 0$

- If β is regime dependent there is no closed-form relation between the above coefficients

- log-linear approximation might be used (Bansal and Zhou (2002))

- Our approach does not require $\boldsymbol{\beta}$ to be constant

- Let $\mathcal{H}_k := \mathcal{F}_T^X \vee \mathcal{F}_k^r$ the enlarged information set generated by histories of the Markov chain up to the maturity time T and the short rate process up to time k

- The conditional price of the zero-coupon bond at time t given $\mathcal{H}_k$ is defined by:

$$\tilde{B}(k,T) := E^Q\Big[\exp\Big(-\sum_{t=k+1}^{T} r_t\Big)|\mathcal{H}_k\Big]\ , \quad k = 0, 1, \cdots, T$$

- **Proposition 2** *The conditional bond price $\tilde{B}(k,T)$ has the following exponential affine form:*

$$\tilde{B}(k,T) = \exp(A_{1k} + A_{2k} r_k) \ , \quad k = 0, 1, \cdots, T \ , \tag{1}$$

where the stochastic processes $\{A_{1k}\}_{k=0,1,\cdots,T}$ and $\{A_{2k}\}_{k=0,1,\cdots,T}$ satisfy the following system of coupled stochastic recursions:

$$\begin{aligned} A_{1T} &= A_{2T} = 0 \\ A_{1,k-1} &= A_{1k} + (A_{2k} - 1)\left(\langle \alpha, \mathbf{X}_{k-1} \rangle + \frac{1}{2}(A_{2k} - 1) \langle \sigma, \mathbf{X}_{k-1} \rangle^2 \right) \\ A_{2,k-1} &= (A_{2k} - 1) \langle \beta, \mathbf{X}_{k-1} \rangle \ , \quad k = 1, 2, \cdots, T \end{aligned}$$

- Prove this result by backward induction

- $\tilde{B}(k,T)$ is a function of $\mathbf{X}_k$, $\mathbf{X}_{k+1}$, $\cdots$, $\mathbf{X}_{T-1}$:

$$\tilde{B}(k,T,\mathbf{X}_k,\mathbf{X}_{k+1},\cdots,\mathbf{X}_{T-1})$$

- Note that A_{1k} and A_{2k} are measurable wrt the tail σ-algebra generated by $\mathbf{X}_k$, $\mathbf{X}_{k+1}$, $\cdots$ and $\mathbf{X}_{T-1}$

- **Proposition 3** *Given $\mathcal{G}_k$, the bond price has the following form:*

$$B(k,T) \;=\; \sum_{i_k,i_{k+1},\cdots,i_{T-1}=1}^{N} \left[\prod_{l=k}^{T-1} c_{i_l i_{l+1}}\right] \tilde{B}(k,T,\mathbf{e}_{i_k},\mathbf{e}_{i_{k+1}},\cdots,\mathbf{e}_{i_{T-1}})$$

- This result is obtained from taking expectation of $\tilde{B}(k,n)$ conditioning on $\mathcal{G}_k$ under $\mathcal{Q}$ and by enumerating transition probabilities of the Markov chain from time k to time $T-1$.

§6. Summary

- Develop a discrete-time, Markov, regime-switching, affine term structure model for valuing bonds and other interest-rate sensitive securities

- Introduce a pricing kernel by the product of two density processes for measure changes, one for the interest rate process and another for the Markov chain

- Control both the market risk and the long-term economic risk

- Provide a simple derivation of a weighted average of exponential-affine forms for zero coupon bond

- Empirical investigation of this pricing model - future work

$\sim$ **Thank you !** $\sim$

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