Welcome

Fourier Transform Methods in Math Finance

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Outline

Introduction

Fourier Transform, Discrete Fourier Transform, and Fast Fourier Transform

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Application in Math Finance

- Basic ideas
- Selected Papers

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Time line

A rough time line of the development of the literature

Definition of Fourier Transform

Definition

Let $f(t)$ be a function defined on $\mathbb{R}$. Given the Fourier Parameters (a, b), the Fourier Transform (**FT**) of f is defined as the function

$$\mathcal{F}(f)(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(t) e^{ib\omega t} dt.$$

The domain of $\mathcal{F}(f)$ consists of real numbers ω that the improper integral converges.

standard properties

time shift, scaling, convolution, etc

Inverse FT and choice of a, b

Inverse Fourier Transform

The inverse Fourier transform, if exists, is given by

$$\mathcal{F}^{-1}(F)(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} F(\omega) e^{-ib\omega t} d\omega$$

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Choice of (a, b)

- classic physics: $(a, b) = (1, 1)$
- modern physics: $(a, b) = (0, 1)$
- pure mathematics and systems engineering: $(a, b) = (1, -1)$
- signal processing: $(a, b) = (0, -2\pi)$
- math finance: $(1, 1)$ or $(0, -2\pi)$

Discrete Fourier Transform

Definition

Discrete Fourier Transform (**DFT**) is a map from $\mathbb{C}^N$ to $\mathbb{C}^N$ that transform $(x_0, x_1, \cdots, x_{N-1}) \in \mathbb{C}^N$ to $(X_0, X_1, \cdots, X_{N-1})$ with

$$X_k = \sum_{j=0}^{N-1} x_j e^{-\frac{2\pi ijk}{N}}.$$

Approximate FT by DFT

Illustration

We know $\int_0^\infty f(t)e^{-i\omega t}dt \approx \int_0^A f(t)e^{-i\omega t}dt$ for large A.
To compute $\int_0^\infty f(t)e^{-i\omega t}dt$ on $[-B, B]$, we can set $\eta = \frac{A}{N-1}$, $\lambda = \frac{2B}{N-1}$ such that $\eta\lambda = \frac{2\pi}{N}$, and let $t_j = j\eta$, $\omega_k = -B + k\lambda$, then

$$\int_0^\infty f(t)e^{-i\omega t}dt \approx \sum_{j=0}^{N-1} e^{-i\omega_k t_j} f(t_j)\eta = \sum_{j=0}^{N-1} e^{-\frac{2\pi ijk}{N}} e^{Bt_j} f(t_j)\eta,$$

hence we can approximate FT by DFT.

Fast Fourier Transform

- Fast Fourier Transform (**FFT**) refers to some efficient algorithms of computing DFT and its inverse.
- Computing DFT directly from the definition requires $O(N^2)$ arithmetical operations, but FFT algorithms can substantially improve the speed of the computation.
- The Cooley-Tukey algorithm, for instance, can compute DFT in $O(N \log N)$ operations. The algorithm uses a divide-and-conquer strategy.

Why Fourier Transform

- More sophisticated financial models are introduced beyond the classic Black-Schole model
- Difficulties in computing expectations in the new models due to the absence of explicit density function
- Fourier transform directly relates characteristic function to the corresponding density function, hence can be naturally implemented in the computation

Carr and Madan (1999) [1]-The original idea

Given an European option, let $k = \ln K$, $s_t = \ln S_t$, and let $q(x)$ be the risk-neutral density function of s_T, then the time zero price of the option is $C(k) = \int_k^\infty e^{-rT}(e^s - e^k)q(x)dx$, then choose $\alpha > 0$ such that $c(k) = e^{\alpha k}C(k)$ (a sufficient condition is $E[S_T^{\alpha+1}] < \infty$) is square-integrable, $C(k)$ can be computed by

$$c(k) \to \psi(v) = \mathcal{F}(c)(v) \to C(k) = e^{-\alpha k}\mathcal{F}^{-1}(\psi)(k),$$

$\psi(v)$ is related to the characteristic function $\phi(v)$ of s_T by

$$\psi(v) = \frac{e^{-rT}\phi(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v},$$

and then FFT methods can be used to compute $C(k)$.

Classification of literature

- Types of derivative: Discrete Asian Options, spread opions, electricity derivatives, Parisian Options, etc.

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- Generalization of methods
- New methods: The COS Method, the CONV method, fractional FFT, etc.
- General discussion
- Other: application in insurance, performance evaluation, error control, etc.

Raible (2000) [2]

- In Chapter 3 of his Ph.D Thesis, Raible derived a formula of prices of European options by bilateral Laplace transform w.r.t. log spot price, and FFT method can be used to evaluate the formula.

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- The Method: Let $c = -\ln S_0$ and let Q be the risk-neutral probability measure, and let $g(S_T)$ be the payoff function of a European option such that $S_t = S_0 e^{X_t}$. Then the price of the option at time 0 is

$$V(c) = e^{-rT} E_Q[g(S_T)] = e^{-rT} E_Q[g(e^{-c+X_T})].$$

Raible (2000) [2]-Continued

Let $\pi(x) = g(e^{-x})$, then

$$V(c) = e^{-rT} E_Q[\pi(c - X_T)] = e^{-rT} \int_{\mathbb{R}} \pi(c - x)\rho(x)dx,$$

which is the convolution of π and ρ. Then by applying the bilateral Laplace transform on V and then the inverse transform we can get

$$V(c) = \frac{e^{-rT+cR}}{2\pi} \int_{\mathbb{R}} e^{icu} \mathcal{L}_{\pi}(R + iu)\phi_{X(T)}(iR - u)du,$$

which can be considered as a Fourier Transform, hence FFT can be applied to its calculation.

Duffie et. al. (2000) [3]

- This paper extends the literature on affine asset-pricing models by deriving analytically tractable pricing relations for a wide variety of valuation problems using transform methods.

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- This paper extends the literature on affine asset-pricing models by deriving analytically tractable pricing relations for a wide variety of valuation problems using transform methods.
- The method: For an option with payoff $(e^{dX_T} - c)_+$ at time T where X_t is an affine jump-diffusion (AJD) process. Let $G_{a,b}(y)$ be the price of a security that pays e^{aX_T} at time T if $bX_T \leq y$ and 0 otherwise. Then the time zero price of the option is

$$p = G_{d,-d}(-\ln c) - cG_{0,-d}(-\ln c).$$

Duffie et. al. (2000) [3]-Continued

To compute p, we can first compute the Fourier transform $\mathcal{G}_{a,b}(z)$ of $G_{a,b}(y)$ and then apply the inverse transform. In many cases $\mathcal{G}_{a,b}(z)$ can be computed explicitly by

$$\mathcal{G}_{a,b}(z) = E_{\chi}\left[\exp(-\int_0^T R(X_s, s)ds)e^{uX_T}\right]$$

for $u = a + ibz$.

Dempster and Hong (2000) [4]

- The paper extend the fast Fourier Transform technique in Carr and Madan (1999) [1] to a multi-factor setting for generic spread options.

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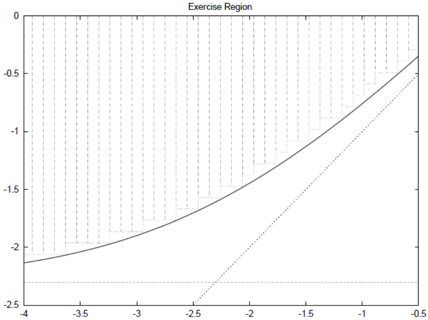
- The paper extend the fast Fourier Transform technique in Carr and Madan (1999) [1] to a multi-factor setting for generic spread options.
- The Carr and Madan approach can be applied to a correlation option (option with payoff $(S_1(T) - K_1)_+(S_2(T) - K_2)_+$) using 2-dimensional FFT. But when it comes to spread option (e.g. option with payoff $(S_1(T) - S_2(T) - K)_+$), there is an obstacle of getting the method work: The exercise region no longer has straight line edges.

Dempster and Hong (2000) [4]-Continued

The paper uses rectangular strips to approximate the exercise region and for each rectangular region the FFT method can be applied. The region $\{(s_1, s_2) \mid \; e^{s_1} - e^{s_2} - K \geq 0\}$:



Lewis (2001) [5]

The paper derives a formula that is similar to that of Raible's, but it consistently uses the generalized Fourier Transform and has more detailed discussion on the strip of integration.

Chourdakis (2004) [6]

- The paper uses the method of fractional FFT to compute option prices and conducts experiments to show the efficiency of the method compared to straight FFT.

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- The paper uses the method of fractional FFT to compute option prices and conducts experiments to show the efficiency of the method compared to straight FFT.
- The Method: The fractional FFT procedure can rapidly compute sums of the form $\sum_{j}^{N-1} x_j e^{-2\pi ijk\alpha}$. The standard FFT can be considered as a special case when $\alpha = 1/N$. Instead of imposing the condition $\eta\lambda = 2\pi/N$, fractional FFT requires $\eta\lambda = \alpha$ and α can be chosen accordingly to the context.

Liu et. al. (2006) [7]

- The paper applies the methodology of Carr and Madan (1999) and develops an FFT method for a model where the underlying price is governed by a regime-switching geometric Brownian motion.

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- The paper applies the methodology of Carr and Madan (1999) and develops an FFT method for a model where the underlying price is governed by a regime-switching geometric Brownian motion.
- Regime-switching model:

$$\frac{dS(t)}{S(t)} = \mu(\alpha(t))dt + \sigma(\alpha(t))d\widetilde{B}(t)$$

where $\alpha(t)$ is a finite-state continuous time Markov chain with state space $M = \{1, 2, \cdots, m\}$.

Lord et. al. (2008) [9]

- The paper propose a method called CONV to price Bermudan options and American options. The method recognizes the well-known risk-neutral valuation formula as a convolution and then evaluate it by FFT.

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- The paper propose a method called CONV to price Bermudan options and American options. The method recognizes the well-known risk-neutral valuation formula as a convolution and then evaluate it by FFT.
- A Bermudan option is an option where the buyer has the right to exercise at a set (always discretely spaced) number of times. This is intermediate between a European option an American option.

Family tree

- log strike: Car and Madan (1999)$\rightarrow$ Duffie et. al. (2000) $\rightarrow$ Lee (2004)
- log spot price: Raible (2000) $\rightarrow$ Lewis (2001) $\rightarrow$ Jackson et. al. (2007)
- My review

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Thank You

Thank you for listening!

Questions?