Capital Requirements and Optimal Investment for Insurance Companies

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Standard Portfolio Theory

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- For a given level of risk, maximize the expected return on the portfolio
- For a given level of expected portfolio return, minimize the risk associated to the portfolio

Risk Measures

$\mathcal{R}$ can be one of the following risk measures:

- The *variance* of the portfolio's return is proposed by Markowitz in 1952
- The *Value at Risk* of a loss random variable Y at a confidence level α, $0 \leq \alpha \leq 1$, is formally defined as

$$VaR_{\alpha}(Y) = \inf\{y \in \mathbb{R} : \Pr(Y \leq y) \geq \alpha\}$$

- The *Conditional Value at Risk* of a loss funtion $f(\mathbf{x}, \mathbf{y})$ at a condifidence level α, $0 \leq \alpha \leq 1$, is proposed by Rockaffellar and Uryasev (2000) formally defined as:

$$CVaR_{\alpha}(Y) = E[Y | Y > VaR_{\alpha}(Y)]$$

Objective

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- a series of optimal investment models for finance and insurance companies are developed.
- an innovative optimal investment model for an insurer is proposed in the literature.

Portfolio optimization in Finance

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- The total wealth at time T is given by:

$$W = \mathbf{x}^{\mathbf{T}}\mathbf{R} = \sum_{i=1}^{n} x_i R_i, \text{ with, } \sum_{i=1}^{n} x_i = 1,$$

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- The standard optimization problem:

$$\begin{aligned} &\min_{\mathbf{x}} \mathcal{R}(W), \text{ subject to} \\ &\mathbf{x}^{\mathbf{T}} E[\mathbf{R}] \ \geq \ \mu, \\ &\mathbf{x}^{\mathbf{T}}\mathbf{1} \ = \ 1, \\ &\mathbf{x} \ \geq \ \mathbf{0}. \end{aligned}$$

Mean-Variance Model

Using the risk measure $\mathcal{R}(W) = Var(W)$, we can derive the mean-variance optimization problem.
The MVO is a convex quadratic problem:

$$\min_{\mathbf{x}\in[\mathbf{0},\mathbf{1}]^{\mathbf{n}}} \mathbf{x}^{\mathbf{T}}[\frac{1}{N}\sum_{k=1}^{N}(\mathbf{R}^k - \mathbf{r})(\mathbf{R}^k - \mathbf{r})]\mathbf{x} \quad \text{subject to} \tag{1}$$

$$\mathbf{x}^{\mathbf{T}}\frac{1}{N}\sum_{k=1}^{N}\mathbf{R}^k \geq \mu,$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{1} = 1,$$

$$\mathbf{x} \geq \mathbf{0}.$$

CVaR nonlinear Model

The nonlinear optimization problem is equivalent to minimizing the following expression:

$$\min_{(\mathbf{x},t)\in\mathbf{X}\times\mathbb{R}} t + \frac{1}{N(1-\alpha)}\sum_{k=1}^{N}[f(\mathbf{x},\mathbf{y}_N)-t]^{+} \quad \text{subject to} \quad (2)$$

$$\mathbf{x}^{\mathbf{T}}\frac{1}{N}\sum_{k=1}^{N}\mathbf{R}^{k} \geq \mu,$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{1} = 1,$$

$$\mathbf{x} \geq \mathbf{0}.$$

CVaR linear Model

Our optimization Problem (2) can be reduced to a linear programming problem. Introducing N dummy variables u_k, $k = 1, \cdots, N$, minimizing $CVaR_\alpha$ ca be replaced by the linear function.

$$\min_{\mathbf{x},\zeta,u_k} \zeta + \frac{1}{N(1-\alpha)} \sum_{k=1}^{N} u_k, \quad \text{subject to} \tag{3}$$

$$f(\mathbf{x}, \mathbf{y}) - \zeta \leq u_k,$$

$$\mathbf{x}^\mathbf{T} \frac{1}{N} \sum_{k=1}^{N} \mathbf{R}^k \geq \mu,$$

$$\mathbf{x}^\mathbf{T} \mathbf{1} = 1,$$

$$\mathbf{x} \geq \mathbf{0},$$

$$u_k \geq 0$$

VaR Model

- Lack of subadditivity
- In the case of a finit scenarios, VaR is non-smooth, nonconvex and multi-extreme
- Larsen, N. and Uryasev, S. “Algorithms for Optimization of VaR”
- To reduce VaR by solving a series of CVaR problem

The Insurance Models in One Period

- Premiums P are received at time 0 and are invested over a fixed period of time of $(0, T)$ in n securities.

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- Premiums P are received at time 0 and are invested over a fixed period of time of $(0, T)$ in n securities.
- The insurer net loss at time T is given by

$$L(\mathbf{x}) = Y - P\sum_{i=1}^{n} x_i R_i, \text{ with } \sum_{i=1}^{n} x_i = 1$$

where Y is the insurer liability assumed to be paid only at time T.

Mean Variance Model

Using the risk measure $\mathcal{R}(W) = Var(W)$, we derive the mean-variance optimization problem:

$$\begin{aligned} \min_{\mathbf{x}\in[\mathbf{0},\mathbf{1}]^{\mathbf{n}}} \quad & \mathbf{x}^{\mathbf{T}}[\frac{1}{N}\sum_{k=1}^{N}(\mathbf{R}^k - \mathbf{r})(\mathbf{R}^k - \mathbf{r})]\mathbf{x} + \frac{1}{N}\sum_{k=1}^{N}(Y_k - \frac{1}{N}\sum_{k=1}^{N} Y_k)^2 \\ \text{s.t.} \quad & \mathbf{x}^{\mathbf{T}}\frac{1}{N}\sum_{k=1}^{N}\mathbf{R}^k \ \geq \ \mu, \\ & \mathbf{x}^{\mathbf{T}}\mathbf{1} \ = \ 1, \\ & \mathbf{x} \ \geq \ \mathbf{0}. \end{aligned} \tag{4}$$

CVaR Model

Using the risk measure $\mathcal{R}(W) = CVaR(W)$, we derive the CVaR optimization problem:

$$\min_{\mathbf{x},t,u_k} CVaR_\beta = \min_{\mathbf{x},t,u_k} t + \frac{1}{N(1-\beta)} \sum_{k=1}^{N} u_k, \quad \text{subject to} \quad (5)$$

$$P\mathbf{x}^{\mathbf{T}} R_i^k - Y_k - t - u_k \leq 0,$$

$$\mathbf{x}^{\mathbf{T}} \frac{1}{N} \sum_{k=1}^{N} \mathbf{R^k} \geq \mu,$$

$$\mathbf{x}^{\mathbf{T}} \mathbf{1} = 1,$$

$$u_k \geq 0,$$

$$\mathbf{x} \geq \mathbf{0}.$$

Definition of Ruin probability

The ruin probability:

$$\begin{aligned}\Pr(L(\mathbf{x}) > 0) &= \int_{\Re_+^n} \Pr(Y > P\mathbf{x}^{\mathbf{T}}\mathbf{R}) \Pr(\mathbf{R} \in d\mathbf{r}) \\ &\approx \frac{1}{N}\sum_{k=1}^{N} \Pr(Y > P\mathbf{x}^{\mathbf{T}}\mathbf{R}^k).\end{aligned}$$

Solvency II Objectives:

Solvency II is known as the base II for insurace markets in the European Union. A key question to be addressed in the first pillar of the Solvency II is the determination of capital requirements for insurance companies.

- Improve the protection of policyholders and beneficiaries
- Create a more harmonised, risk-orientated solvency regime resulting in risk-adjusted capital requirements

Economic Capital

- The "Economic Capital" (C) is closely connected to the Value at Risk concept which is very popular in finance, and the economic capital does in fact originate from the VaR.
- Holding more capital increases the company's capacity. However, excessive equity capital reduces competitiveness because of high equity cost and some market friction.

$$C = \inf_{x} \{\Pr(L \geq x) = 1 - \alpha\}$$

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Objective: minimize the capital and find the optimal portfolio allocation.

NonConvex Optimization Problems

Therefore, minimizing $\mathrm{VaR}_\alpha(Y)$ with nonConvex contraints problems is equivalent to:

$$\min_{C \geq 0, \mathbf{x} \in [0,1]^n} C \quad \text{subject to} \tag{6}$$

$$\frac{1}{N}\sum_{k=1}^{N} \Pr\left(Y > (P+C)\mathbf{x}^{\mathbf{T}}\mathbf{R}^k\right) \leq 0.5\%,$$

$$\mathbf{x}^{\mathbf{T}}\mathbf{1} = 1.$$

We derived Problem (6) into convex optimization problems by substitute $D = C + P$ and $\mathbf{y} = D\mathbf{x}$.

Convex Optimization Problems

$$\min_{D \geq P, \mathbf{y} \in [0,D]^n} D - P, \quad \text{subject to} \tag{7}$$

$$\frac{1}{N} \sum_{k=1}^{N} \Pr\left(Y > \mathbf{y}^{\mathbf{T}} \mathbf{R}^k\right) \leq 0.5\%,$$

$$\mathbf{y}^{\mathbf{T}} \mathbf{1} = D.$$

Example

Objective: To test how good the ruin problability approximation is.

Example: Consider a portfolio formed by one riskless asset and risky asset. We assume a risk-free rate of return of 4% and the gross return on the risky asset, R, having a $Normal(\mu = 1.14, \sigma = 0.2)$ distribution. The insurance company receives a premium of $1,100$. The incurred loss random variable, Y, is assumed to be:

$Y \sim Normal(\mu_Y = 1000, \sigma_Y = 150)$

Theoretical Opotimization Problem

The Theoretical Optimization Problem is:

$$\min_{C\geq 0,\mathbf{x}\in[0,1]^n} C \quad \text{subject to} \tag{8}$$

$$\frac{E[Y]-(P+C)\mathbf{x}^{\mathbf{T}}\mathbf{R}}{\sqrt{\sigma_Y^2+\sigma^2(P+C)^2x_2^2}} \geq \Phi(0.5\%)^{-1},$$

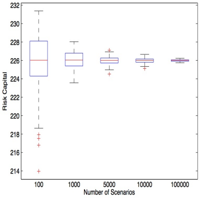
$$\mathbf{x}^{\mathbf{T}}\mathbf{1} = 1.$$

where Φ is the CDF of the standard normal distribution, $\mathbf{x}^{\mathbf{T}} = (x_1, x_2)$, and $\mathbf{r} = (1.04, \mu)'$.

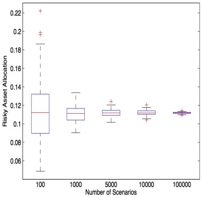
Average minimum capital required and optimal weights based on simulated data - 2 assets

Number of Scenarios	C^*	x^*
100	225.66 (3.3521)	0.1143 (0.0329)
1,000	226.02 (0.8360)	0.1111 (0.0085)
5,000	225.97 (0.4550)	0.1120 (0.0043)
10,000	225.98 (0.2971)	0.1120 (0.0030)
100,000	225.99 (0.0987)	0.1119 (0.0009)
Theoretical	225.99	0.1119

Capital Required and Risky Asset allocations for Different Scenarios.



(a) Normal Capital



(b) Normal Allocation

Aggregation to Insurance Risk

- Portfolio 1: $Y_1 \sim \text{Pareto}(\alpha, \lambda_1 = 3000)$;
- Portfolio 2: $Y_2 \sim \text{Pareto}(\alpha, \lambda_2)$;
- Portfolio 1&2: Joint losses $Z = Y_1 + Y_2$.

Objective: To compare the capital required in portfolio 1&2 and the sum of that in portfolio 1 and portfolio 2 if we combined the insurance losses.

Definition

The joint survival function of Y_1 and Y_2:

$$S(Y_1, Y_2; \lambda_1, \lambda_2) = \left(\frac{y_1}{\lambda_1} + \frac{y_2}{\lambda_2} + 1\right)^{-\alpha}$$

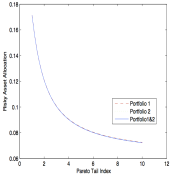
Definition

Definition

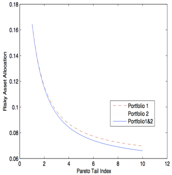
The survival function the of the sum of Y_1 and Y_2, i.e., $Z = Y_1 + Y_2$, has the tail given by:

$$S(Z;\lambda_1,\lambda_2) = \begin{cases} \lambda^{\alpha}(z+\lambda_1)^{-\alpha-1}(\alpha z + z + \lambda_1) & \text{if } \lambda_1 = \lambda_2, \\ \frac{\lambda_1\left(\frac{\lambda_1}{z+\lambda_1}\right)^{\alpha} - \lambda_2\left(\frac{\lambda_2}{z+\lambda_2}\right)^{\alpha}}{\lambda_1 - \lambda_2} & \text{if } \lambda_1 > \lambda_2. \end{cases}$$

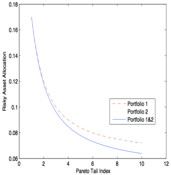
Risky asset Allocations vs. Pareto Tail Index: Problem (7) with various values of λ



(c) $\lambda_2 = 100$



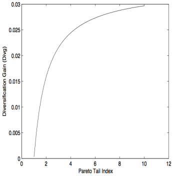
(d) $\lambda_2 = 1{,}000$



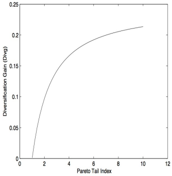
(e) $\lambda_2 = 3{,}000$

Diversification Gain: Problem (7) with various values of tail index paramter α

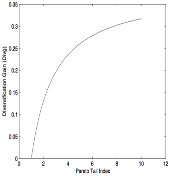
$$\text{Diversification Gain} = 1 - \frac{C_{12}^*}{C_1^* + C_2^*}$$



(f) $\lambda_2 = 100$



(g) $\lambda_2 = 1{,}000$



(h) $\lambda_2 = 3{,}000$

Return on Capital

- Shareholders want more Return on Capital;
- Invest into the market;
- Higher risk $\Rightarrow$ Higher return;

Return on Economic Capital:

$$E[ROC] = \frac{(P+C)\mathbf{x}^{\mathbf{T}}\mathbf{r} - \mathbf{E}[Y]}{C};$$

Algorithm 1

Algorithm 1: Minimization of the Capital Required with a Certain Level Ruin Probability and Expected Return on Capital Constraint.

$$\min_{C \geq 0, \mathbf{x} \in [0,1]^n} C \quad \text{subject to} \tag{9}$$

$$\frac{1}{N} \sum_{k=1}^{N} \Pr\left(Y > (P+C)\mathbf{x}^{\mathbf{T}}\mathbf{R}^k\right) \leq 0.5\%,$$

$$\frac{(P+C)\mathbf{x}^{\mathbf{T}}\mathbf{r} - \mathbf{E}[Y]}{C} \geq \mu,$$

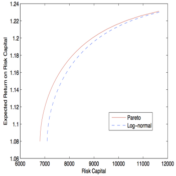
$$\mathbf{x}^{\mathbf{T}}\mathbf{1} = 1.$$

Algorithm 2

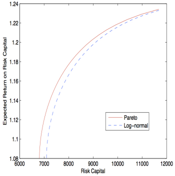
Algorithm 2: Maximization of the Expected Return on Capital with a Certain Level Ruin Probability and Capital Constraint.

$$
\begin{aligned}
\max_{C\geq 0, \mathbf{x}\in[0,1]^n} E[ROC] &= \min_{C\geq 0, \mathbf{x}\in[0,1]^n} \frac{C}{(P+C)\mathbf{x}^{\mathbf{T}}\mathbf{r} - \mathbf{E}[Y]} \\
\text{s.t.} \quad \frac{1}{N}\sum_{k=1}^{N} \Pr\left(Y > (P+C)\mathbf{x}^{\mathbf{T}}\mathbf{R}^k\right) &\leq 0.5\%, \qquad (10) \\
C &\leq \nu, \\
\mathbf{x}^{\mathbf{T}}\mathbf{1} &= 1.
\end{aligned}
$$

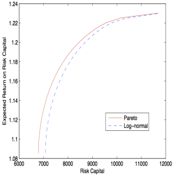
Efficient frontier: Convex Optimal Problem with various ρ at CL of 99.5%



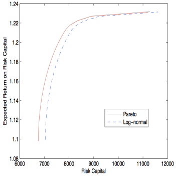
(i) $\rho = 1$



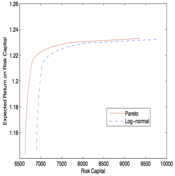
(j) $\rho = 0.75$



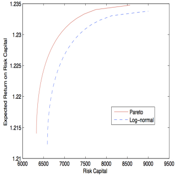
(k) $\rho = 0.25$



(l) $\rho = -0.25$

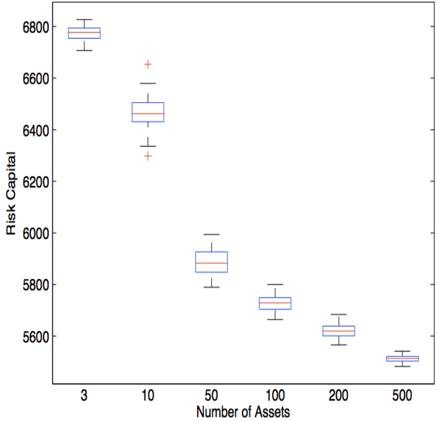


(m) $\rho = -0.75$



(n) $\rho = -1$

Capital Required with Different Number of Assets



Key Findings

- Mean variance, VaR, and CVaR optimization approaches
- Optimal assets allocation in finance field and insurance field
- New approach: minimize Capital required subject to a certain level of ruin probablity
- Advantages: convex, subadditive, etc

What I learnt

- Simulation and programming skills
- Optimization toolbox in Matlab
- Latex techniques

Acknowledgements

Thank you, committee.

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