Long-range dependence and non-semimartingale models in finance

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Definition of Fractional Brownian Motion

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition 1. The (two-sided, normalized) fractional Brownian motion $($fBm$)$ with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, P)$, having the properties

(i) $B_0^H = 0$,

(ii) $EB_t^H = 0, t \in \mathbb{R}$,

(iii) $EB_t^H B_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), s, t \in \mathbb{R}$.

Remark 2. Since $E(B_t^H - B_s^H)^2 = |t-s|^{2H}$ and B^H is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.

Representation of fBm via the Wiener Process on a Finite Interval

Sometimes it is convenient to consider a one-sided fBm $B^H = \{B_t^H, t \geq 0\}$ and to represent it as a functional of the form $B_t^H = \varphi(W_s, 0 \leq s \leq t)$, of some Wiener process $W = \{W_t, t \geq 0\}$. For this purpose consider the kernel

$$m_H(t,s) = C_H\Big((\frac{t}{s})^{\alpha}(t-s)^{\alpha} - \alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^{\alpha}du\Big),$$

where

$$C_H = \left(\frac{2H\Gamma(1-\alpha)}{\Gamma(1-2\alpha)\Gamma(\alpha+1)}\right)^{\frac{1}{2}},$$

and $\alpha = H - \frac{1}{2}, \quad H \in (0,1)$.

Then one has the representation

$$B_t^H = \int_0^t m_H(t,s)dW_s. \tag{1}$$

Arbitrage in the “Pure” Fractional Model.

Results of Shiryaev and Dasgupta

Consider a $(B(r), S(r))$-market with

$$\begin{aligned} &B_t(r) = e^{rt}, \\ &S_t(r) = e^{\mu t + \sigma B_t^H}, \quad t \geq 0, \qquad H \in (1/2, 1). \end{aligned} \tag{2}$$

Let for simplicity $\mu = r, \ \sigma = 1$. We construct a portfolio $\pi = (\beta, \gamma)$ with $\beta_t = 1 - e^{2B_t^H}$, $\gamma_t = 2(B_t^H - 1)$. For such a portfolio we have that the corresponding capital X_t^π equals

$$X_t^\pi = \beta_t B_t(r) + \gamma_t S_t(r) = e^{rt} \left(e^{B_t^H} - 1 \right)^2 .$$

From Itô formula for pathwise integral w.r.t. fBm,

$$X_t^\pi = \int_0^t e^{rs} \left(e^{B_s^H} - 1 \right) \left(r\, ds + 2e^{B_s^H} dB_s^H \right) = \int_0^t \beta_s dB_s(r) + \int_0^t \gamma_s dS_s(r), \tag{3}$$

and (3) exactly means that the strategy π is self-financing strategy in usual sense. So, for this portfolio $X_0^\pi = 0$ and $X_t^\pi > 0$ a.s. for any $t > 0$, and everyone understands that it is an arbitrage possibility (in any appropriate definition). This is Shiryaev’s example (2001).

A very close result was obtained by Dasgupta (1998). He considers a one-dimensional portfolio $\pi_t, 0 \leq t \leq 1$, the same model as in (2), defines discounted gain as

$$G_t = \int_0^t \pi(s) B_s^{-1}(r) \left(\sigma dB_s^H + (\mu - r) ds \right),$$

and determines arbitrage as the following possibility:

(a) there exists $\alpha \in \mathbb{R}$ such that $P\{G_t \geq \alpha, 0 \leq t \leq 1\} = 1;$

(b) $P\{G_t \geq 0\} = 1, \quad (c)$ $P\{G_1 > 0\} > 0.$

Now, consider the particular case $\mu = r$ and the particular portfolio

$$\pi_t = 2e^{rt+\sigma B_t^H} \left(e^{\sigma B_t^H} - 1 \right). \tag{4}$$

With portfolio (4) the gain process equals

$$G_t = \int_0^t 2e^{\sigma B_s^H} \left(e^{\sigma B_s^H} - 1 \right) \sigma dB_s^H = \left(e^{\sigma B_t^H} - 1 \right)^2.$$

Of course, we obtain arbitrage possibility. As a conclusion, we see that the "pure" continuous-time model based on fBm is not arbitrage-free, if the arbitrage possibility is defined in any appropriate terms. The same fact is emphasized in PhD thesis of Cheridito (2001), the paper of Salopek (1998).

Now we can discuss discrete-time models and “mixed” models (the latter ones are much more promising).

Mixed Brownian - Fractional Brownian Model:
Absence of Arbitrage and Related Topics

Let $\{W_t, t \geq 0\}$ be a standard Wiener process and $\{B_t^H, t \geq 0\}$ be a fBm with the Hurst index $H \in (1/2, 1)$, both defined on a filtered probability space $\big(\Omega,\, \mathcal{F},\, \{\mathcal{F}_t,\, t \geq 0\},\, P\big)$.

Consider a mixed version of the Black-Merton-Scholes model, i.e. a $\big(B, S\big)$-market with a bond B and a stock S, where

$$B_t = e^{rt}, \quad S_t = e^{aW_t + bB_t^H + ct}, \quad r, a, b, c \in \mathbb{R},\ t \in \mathbb{R}_+. \tag{5}$$

For a given strategy (or a portfolio) $\pi = \{\beta_t, \gamma_t, t \geq 0\}$ the capital $\{X_t, t \geq 0\}$ corresponding to this portfolio equals

$$X_t = B_t \cdot \beta_t + S_t \cdot \gamma_t. \tag{6}$$

We make the following assumptions about the strategy π:

1) π is self-financing strategy, i.e.

$$X_t = X_0 + \int_0^t \beta_s \, dB_s + \int_0^t \gamma_s \, dS_s; \tag{7}$$

2) π is a strategy of Markov type, i.e.

$$\beta_t = \beta\big(S_t, t\big), \quad \gamma_t = \gamma\big(S_t, t\big). \tag{8}$$

One needs to be accurate with condition (7), for it to reflect the real economical concept of "self-financing". This entails that the meaning of the second integral in (7) should be specified clearly. We understand it now in the path-wise sense, i.e. as the following limit with probability 1:

$$\int_0^t \gamma_s \, dS_s = \lim_{\max|s_{k+1}-s_k|\to 0} \sum_{k=0}^{n-1} \gamma_{s_k} \left(S_{s_{k+1}} - S_{s_k}\right).$$

Here, the sum $\sum_{k=0}^{n-1} \gamma_{s_k} \left(S_{s_{k+1}} - S_{s_k}\right)$ is an obvious formula for the capital, earned on the price variation of S with a piecewise buy-and-hold strategy $\{\widetilde{\gamma}_t, t \in \mathbb{R}_+\} = \{\gamma_{s_k}, \ s_k \le t < s_{k+1}, \ t \ge 0\}$. Hence, the integral $\int_0^t \gamma_s \, dS_s$, as the capital earned on S with the continuous strategy $\{\gamma_t, t \in \mathbb{R}_+\}$, agrees with the "fundamental moral" in the definition of self-financing conditions.

We say that the strategy π is an arbitrage opportunity if there exists $T > 0$ such that

$$X_0 = 0, \quad X_T \geq 0\,(P - a.s.), \quad P(X_T > 0) > 0.$$

In the mixed model (5) with $a \neq 0$ and $b \neq 0$, some results in this direction have been obtained in the papers of Kuznetsov (1999), Cheridito (2001), Mishura and Valkeila (2002), Zähle (2002). More exactly, Kuznetsov established the absence of arbitrage under the condition of independence of processes W and B^H. Cheridito proved that, for $H \in (3/4, 1)$, the mixed model with independent W and B^H is equivalent to the one with Brownian motion and hence it is arbitrage-free. Zähle proved the absence of arbitrage in the general mixed model with independent Wiener process and the process of zero quadratic variation. In the mixed model, studied in the paper of Mishura and Valkeila, there is no requirement of independence. Conversely, the absence of arbitrage is demonstrated under the condition that the process B^H is related to the process W by equation (1).

The main result of this subsection is that the mixed market is arbitrage-free without any conditions on the dependence of W and B^H, if we restrict ourselves to the self-financing Markov-type strategies with smooth β and γ.

Conditions of Self-Financing and Their Consequences

Note that in the case of Markov-type strategy (8), the process of capital X_t can be written as a function of price of the stock S at the moment t:

$$X_t = \Phi\big(S_t, t\big), \tag{9}$$

where

$$\Phi(x,t) = e^{rt} \cdot \beta(x,t) + x \cdot \gamma(x,t). \tag{10}$$

We prove first that the self-financing assumption restricts the class of possible functions Φ in (9).

In the case of $\gamma_t = \gamma(S_t, t)$ with smooth $\gamma(\cdot,\cdot)$, the integral $\int_0^t \gamma_s \, dS_s$ exists and

$$\int_0^t \gamma_s \, dS_s = \int_0^t a \, \gamma_s S_s \, dW_s + \int_0^t b \, \gamma_s S_s \, dB_s^H + \int_0^t \left(c + \frac{a^2}{2} \right) \gamma_s S_s \, ds, \tag{11}$$

where the first integral on the right-hand side is the Itô integral, the second integral is the path-wise Riemann–Stieltjes integral and the third one is the Riemann integral. Formula (11) gives the Itô formula for an exponent of the mixed process.

The Itô integral in (11) appears due to the choice of the left end-point s_k in the expression under the sign of sum in (10). Such a choice is crucial for condition (7) to have the economical sense of self-financing. The second integral $\int_0^t b \, \gamma_s S_s \, dB_s^H$ does not depend on the choice of inner points of the intervals.

Theorem 3. *Let the* $\big(B, S\big)$*-market be given by* (5) *with* $a \neq 0$*. Suppose also that for all* $t > 0$ *the support of the distribution of* S_t *coincides with*

$$\operatorname{supp}(S_t) = [0, +\infty). \tag{12}$$

Then in the class of Markov-type strategies (8) *with*

$$\big\{\beta(x,t), \gamma(x,t)\big\} \subset C^{\,2}\big((0,+\infty)\big) \times C^{\,1}\big([0,+\infty)\big)$$

the condition of self-financing (7) *is equivalent to the following one:*

(i) There exists a function $\phi(x,t) \in C^{\,2}\big((0,+\infty)\big) \times C^{\,1}\big([0,+\infty)\big)$*, which satisfies the equation*

$$\phi'_t(x,t) + \frac{a^2}{2}\, x^2\, \phi''_{xx}(x,t) + r\, x\, \phi'_x(x,t) - r\, \phi(x,t) = 0, \tag{13}$$

and the strategy (β, γ) *can be expressed in terms of* ϕ*:*

$$\begin{cases} \beta(x,t) = e^{-rt}\big(\phi(x,t) - x \cdot \phi'_x(x,t)\big); \\ \gamma(x,t) = \phi'_x(x,t). \end{cases} \tag{14}$$

Remark 4. Condition (12) holds, for example, in the case when the processes W and B^H are jointly Gaussian, and, hence, $\log(S_t) = aW_t + bB_t^H + ct$, $t \geq 0$ is a Gaussian process.

Remark 5. Under condition (i) we have the identity $\Phi(x,t) = \phi(x,t)$.

Remark 6. Let the process $\{Z_t,\, t \geq 0\}$ be defined on $\big(\Omega,\, \mathcal{F},\, \{\mathcal{F}_t, t \geq 0\},\, P\big)$ with $Z_0 = 0$ and $[Z] \equiv 0$, where $[Z]$ stands for usual bracket, i.e. quadratic variation. Then it is not hard to see that Theorem 3 is valid for the $\big(B, \widetilde{S}\big)$-market with

$$B_t = e^{rt}, \quad \widetilde{S}_t = e^{aW_t + Z_t + ct},$$

if only condition (12) holds for the process $\widetilde{S}$.

Absence of Arbitrage

Theorem 7. *Let the* $\big(B, S\big)$*-market be given by* (5) *with* $a \neq 0$*. Let the support of the distribution of* S_t *coincides with*

$$\operatorname{supp}(S_t) = [0, +\infty) \tag{15}$$

for all $t > 0$*.*

Then there is no arbitrage strategy in the class of self-financing Markov-type strategies (8) *with*

$$\big\{\beta(x,t), \gamma(x,t)\big\} \subset C^{\,2}\big((0,+\infty)\big) \times C^{\,1}\big([0,+\infty)\big).$$

Convergence of Lebesgue–Stieltjes Integrals to the Integral w.r.t. fBm

We define an approximation of fBm by

$$B_t^{H,\beta} = \int_0^t s^\alpha dY_s^\beta, \quad t \geq 0, \quad H \in (1/2, 1), \tag{16}$$

where

$$Y_t^\beta := C_H (1-\alpha)^{1/2} \alpha \int_0^t \left(\int_0^{(s-\beta)_+} (s-u)^{\alpha-1} u^{-\alpha} dW_u \right) ds.$$

Then we prove a theorem which establishes the convergence in probability of integrals with respect to $B^{H,\beta}$ from (16) to the integral with respect to fBm.

Theorem 8. *Let the process f is such that for some $\varepsilon > 0$ and for a.a. $\omega \in \Omega$*

$$f(\cdot, \omega) \in C^{\,2(1-H)+\varepsilon}[0, T]. \tag{17}$$

Then

$$\int_0^T f(u)\, dB_u^{H,\beta} \xrightarrow{P} \int_0^T f(u)\, dB_u^H \quad \textit{as} \quad \beta \to 0+,$$

where $\xrightarrow{P}$ denotes the convergence in probability.

The Capital Process as a Limit of Semimartingales

Let the $\big(B,S\big)$-market be given by (5) and a Markov-type strategy $(\widetilde{\beta},\widetilde{\gamma})$ is self-financing for this market. Then the capital, based on this strategy, equals to

$$X_t = X_0 + \int_0^t \widetilde{\beta}\big(S_s, s\big)\, dB_s + \int_0^t \widetilde{\gamma}\big(S_s, s\big)\, dS_s.$$

For $\beta > 0$ and the given $\big(\beta(\cdot,\cdot),\gamma(\cdot,\cdot)\big)$ consider the processes

$$S_t^{\beta} = e^{aW_t + bB_t^{H,\,\beta} + ct} \quad \text{and}$$

$$X_t^{\beta} = X_0 + \int_0^t \widetilde{\beta}\big(S_s^{\beta}, s\big)\, dB_s + \int_0^t \widetilde{\gamma}\big(S_s^{\beta}, s\big)\, dS_s^{\beta}. \tag{18}$$

The Itô formula and definition of $B^{H,\,\beta}$ imply that the process X^{β} can be rewritten as

$$X_t^{\beta} = X_0 + \int_0^t \Big(rB_s\widetilde{\beta}(S_s^{\beta}, s) + (b\,(B_s^{H,\,\beta})_s' + c)S_s^{\beta}\gamma\big(S_s^{\beta}, s\big)\Big)ds + a\int_0^t S_s^{\beta}\gamma\big(S_s^{\beta}, s\big)dW_s, \tag{19}$$

which means that X^{β} is a semimartingale at least if the condition holds

$$\int_0^T E\,\big(S_s^{\beta}\,\widetilde{\gamma}\big(S_s^{\beta}, s\big)\big)^2\, ds < \infty. \tag{20}$$

Theorem 9. *Let* $H \in (3/4, 1)$ *and the pair* $\big(\widetilde{\beta}(\cdot,\cdot), \widetilde{\gamma}(\cdot,\cdot)\big)$ *satisfy the assumptions*

(ii) $\forall\, t \geq 0 \quad \widetilde{\beta}(\cdot, t),\ \widetilde{\gamma}(\cdot, t) \in C^1(\mathbb{R})$

(iii) $\forall\, T, L > 0$ *there exists* $K = K(T, L) > 0$ *such that*

$$\left|\widetilde{\beta}(x,t) - \widetilde{\beta}(x,s)\right| + \left|\widetilde{\gamma}(x,t) - \widetilde{\gamma}(x,s)\right| \leq K\ |t-s|^{\frac{1}{2}}, \quad \forall\ |x| \leq L,\ t, s \in [0, T].$$

(iv) $\forall\, T > 0$ *there exist* $M = M(T) > 0$ *and* $N = N(T) > 0$ *such that*

$$\left|\widetilde{\beta}'_x(x,t)\right| + \left|\widetilde{\gamma}'_x(x,t)\right| \leq M\,(1 + |x|^N), \quad \forall\, t \in [0, T].$$

Then $X_t^\beta \xrightarrow{P} X_t$ *as* $\beta \to 0+$ *for any* $t \in [0, T]$*.*

Remark 10. Evidently, the conditions $(ii) - (iv)$ imply (20) and the pair $\big(B, S^\beta\big)$ can be regarded as a new stock market with a price of the stock being a semimartingale. We have that $S_t^\beta \xrightarrow{P} S_t$ as $\beta \to 0+$ at any moment $t \geq 0$. If, additionally, condition (12) holds for S^β and $\widetilde{\beta}, \widetilde{\gamma} \in (C^{\,2} \times C^{\,1})(\mathbb{R}_+)$, then the strategy $\big(\widetilde{\beta}(S_s^\beta, s), \widetilde{\gamma}(S_s^\beta, s)\big)$ is self-financing and the market $\big(B, S^\beta\big)$ is arbitrage-free. In this case the process X^β is a process of capital in this market.

Equilibrium of financial market. The fractional Burgers equation

The Girsanov Theorem

Consider SDE

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \sigma_1 \int_0^t X_s dW_s + \sigma_2 \int_0^t X_s dB_s^H, t \in [0, T], \tag{21}$$

and suppose that W is underlying Wiener process for B^H and that the coefficient $b(t, x)$ can be presented as $b(t, x) = e(t, x)x$, where $e \in C_b(\mathbb{R}_+ \times \mathbb{R})$. Denote $\hat{e}(t, x) := e(t, x)t^{-\alpha}$. Now we try to change the measure P for another probability measure Q such that $Q_T \ll P_T$, where $P_T := P|_{\mathcal{F}_T}$, $Q_T := Q|_{\mathcal{F}_T}$, and such that the drift $e(t, X_t)X_t dt$ will be annihilated under Q_T. At first, let some probability measure $\widetilde{Q}$ satisfy the assumptions

$$\left.\frac{d\widetilde{Q}}{dP}\right|_{\mathcal{F}_T} = \exp\left\{\int_0^T \varphi_s dW_s - \frac{1}{2}\int_0^T \varphi_s^2 ds\right\}$$

and

$$E \exp\left\{\int_0^T \varphi_s dW_s - \frac{1}{2}\int_0^T \varphi_s^2 ds\right\} = 1 \tag{22}$$

with $E \int_0^T \varphi_s^2 ds < \infty$.

Then from the Girsanov theorem the process $W_t - \int_0^t \varphi_s \, ds =: \hat{W}_t$ will be a Wiener process under the measure $\widetilde{Q}_T$.

Now define

$$l_H(t,s) = C'_H s^{-\alpha}(t-s)^{-\alpha} I_{\{0<s<t\}},$$

where

$$C'_H = \left(\frac{\Gamma(2-2\alpha)}{2H\Gamma(1-\alpha)^3\Gamma(1+\alpha)} \right)^{\frac{1}{2}} .$$

Let the measure $\bar{Q}$ be such that

$$\left. \frac{d\bar{Q}}{dP} \right|_{\mathcal{F}_T} = \exp\left\{ L_T - \frac{1}{2}\langle L \rangle_T \right\},$$

and

$$E \exp\left\{ L_T - \frac{1}{2}\langle L \rangle_T \right\} = 1, \qquad (23)$$

where $L_t = \int_0^t s^\alpha \delta_s dW_s$, $M_t^H = \int_0^t l_H(t,s) dB_s^H$, $W_t = \widehat{\alpha} \int_0^t s^\alpha dM_s^H$,
$\int_0^t l_H(t,s)\psi_s ds = \widetilde{\alpha} \int_0^t \delta_s ds$, $t > 0$ with $E \int_0^t s^{2\alpha} \delta_s^2 ds < \infty$, $\int_0^t |\delta_s| \, ds < \infty$, P-a.s., $t > 0$;
$\widetilde{\alpha} = (1-\alpha)^{1/2}$.

Then the process $\widehat{B}_t^H := B_t^H - \int_0^t \psi_s ds$ is a fBm w.r.t. to measure $\bar{Q}\Big|_{\mathcal{F}_T}$. Now we need in the equality $\widetilde{Q}\Big|_{\mathcal{F}_T} = \bar{Q}\big|_{\mathcal{F}_T} = Q|_{\mathcal{F}_T}$. Hence, in particular, $L_t = \int_0^t \varphi_s dW_s$, whence $\varphi_s = s^\alpha \delta_s$. Therefore we want to find φ and ψ in such a way that common drift equals

$$\sigma_1 \varphi_t + \sigma_2 \psi_t = -e(t, X_t), \quad t \in [0, T]. \tag{24}$$

Now we apply the Abel rearrangement to the relation

$$\int_0^t l_H(t,s)\psi_s ds = \widetilde{\alpha} \int_0^t \delta_s ds = \widetilde{\alpha} \int_0^t s^{-\alpha} \varphi_s ds :$$

$$C'_H \int_0^t (t-u)^{\alpha-1} \int_0^u (u-s)^{-\alpha} s^{-\alpha} \psi_s ds \, du$$

$$= \widetilde{\alpha} \int_0^t (t-u)^{\alpha-1} \int_0^u s^{-\alpha} \varphi_s ds \, du,$$

or

$$B(\alpha, 1-\alpha) C'_H \int_0^t s^{-\alpha} \psi_s ds = \widetilde{\alpha} \int_0^t \frac{(t-u)^\alpha}{\alpha} u^{-\alpha} \varphi_u du,$$

whence after differentiation

$$(\alpha C_H)^{-1} t^{-\alpha} \psi_t = \widetilde{\alpha} \int_0^t (t-u)^{\alpha-1} u^{-\alpha} \varphi_u du. \tag{25}$$

Substituting (25) into (24), we obtain that

$$\sigma_1 \varphi_t + \sigma_2 C''_H t^{\alpha} \int_0^t (t-u)^{\alpha-1} u^{-\alpha} \varphi_u du = -e(t, X_t), C''_H = \alpha C_H \widetilde{\alpha}. \tag{26}$$

Denote $\theta_t := t^{-\alpha}\varphi_t$, then

$$\sigma_1 \theta_t + \sigma_2 C''_H \int_0^t (t-u)^{\alpha-1} \theta_u du = -\hat{e}(t, X_t). \tag{27}$$

The equation (27) is Volterra equation with weak singularity, and its unique solution has a form

$$\theta_t = -\frac{\hat{e}(t, X_t)}{\sigma_1} - \frac{1}{\sigma_1} \int_0^t \sum_{n=1}^{\infty} \rho^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)} \hat{e}(s, X_s) ds,$$

where $\rho = \sigma_2 C''_H \Gamma(\alpha)$. Now we must check the conditions (22) and (23). Evidently, it is sufficient to check Novikov's condition: $E \exp\left\{\frac{1}{2}\int_0^T \varphi_t^2 dt\right\} < \infty$ and $E \exp\left\{\frac{1}{2}\langle L\rangle_T\right\} < \infty$.

But $\varphi_t = -\frac{e(t,X_t)}{\sigma_1} - \frac{1}{\sigma_1} t^\alpha \int_0^t \sum_{n=1}^\infty \rho^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)} \hat{e}(s, X_s) ds$ and is bounded since e is bounded. Further, $\delta_s = \widetilde{\alpha} s^{-\alpha} \varphi_s$, and Novikov's condition evidently holds for the function L, too. So, we proved the following result.

Theorem 11. *Under our suppositions the equation* (21) *under measure Q obtains the differential form*

$$dX_t = \sigma_1 X_t d\widehat{W}_t + \sigma_2 X_t d\widehat{B}_t^H, \quad X(0) = X_0,$$

and its solution has a form

$$X_t = X_0 \exp\{\sigma_1 \widehat{W}_t + \sigma_2 \widehat{B}_t^H - 1/2\sigma_1^2 t\}.$$

Definition 12. The financial market described by the equation (5) is in equilibrium on $[0, T]$ if both the kernel φ_t and likelihood ratio $\frac{dQ}{dP}\Big|_{\mathcal{F}_t}$ are the functions of t and W_t, twice differential in both the variables, and do not depend on the path of $\{W_s, 0 \le s < t\}$.

This definition generalizes the usual definition of equilibrium of the financial market involving only the Wiener process, where path's independence of $\frac{dQ}{dP}\Big|_{\mathcal{F}_t}$ is declared, and the kernel φ_t equals simply $e(t, W_t)$, up to a constant multiplier.

Theorem 13. *If the financial market is in equilibrium, then φ_t satisfies the Burgers' equation*

$$-\varphi(s,x)\varphi_x^{'}(s,x) = \varphi_t^{'}(s,x) + \frac{1}{2}\varphi_{xx}^{''}(s,x).$$

Different Forms of Black–Scholes Equation on the Fractional Market

The Black–Scholes Equation for the Mixed Brownian-Fractional-Brownian Model

Consider a mixed version of the Black–Merton–Scholes model (5) with the value process X_t, described by (6), and self-financing strategies, defined by (7)-(8). Consider $C(t, S_t)$, the price of European call option with striking price K at time $t \in [0, T]$. Suppose that $C \in C^1[0,T] \times C^2(\mathbb{R})$, then we can present the function $\widetilde{C}(t, S(t)) := C(T-t, S(t))$ according to the Itô formula as

$$\begin{aligned} \widetilde{C}(t, S(t)) &= \widetilde{C}(0, x) + \int_0^t \Big(\widetilde{C}'_t(u, S_u) + c\widetilde{C}'_S(u, S_u) S_u + \widetilde{C}'_S \frac{a^2}{2} S_u \\ &+ C''_{ss} \frac{a^2}{2} S_u^2 \Big) du + a \int_0^t \widetilde{C}'_S(u, S_u) S_u dW_u + b \int_0^t C'_S(u, S_u) S_u dB_u^H . \end{aligned} \tag{28}$$

Now, let the portfolio on value process consist of one option and an amount of $-\delta$ of underlying assets. The number $-\delta$ will be specified later. The value of this portfolio equals $X = \widetilde{C} - \delta S$.

The jump in the value of this portfolio in one-step time equals

$$\begin{aligned} dX = d\widetilde{C} - \delta dS = \Big(\widetilde{C}'_t + c\widetilde{C}'_S + \frac{a^2}{2}\widetilde{C}''_{SS}S^2\Big)du + a\widetilde{C}'_S S dW_u + bC'_S S dB^H_u \\ -\delta\Big(aSdW_u + bSdB^H_u + \frac{a^2 S}{2}du + cSdu\Big). \end{aligned} \tag{29}$$

If we choose $\delta = \frac{\partial \widetilde{C}}{\partial S}$ to eliminate the stochastic noise, then

$$dX = \Big(\widetilde{C}'_t + \frac{a^2}{2}\widetilde{C}''_{SS} \cdot S^2\Big)du.$$

The return of an amount X invested in bank account equals $rXdt$ at time dt. For absence of arbitrage, these values must be the same. Hence we obtain the traditional Black–Scholes equation

$$\widetilde{C}'_t + \frac{1}{2}a^2S^2\frac{\partial^2\widetilde{C}}{\partial S^2} - r\widetilde{C} + rS\widetilde{C}'_S = 0,$$

or, in terms of $C(t, S_t)$,

$$-C'_t + \frac{1}{2}a^2S^2\frac{\partial^2 C}{\partial S^2} - rC + rSC'_S = 0.$$

Remark 14. The same equation was obtained by Zähle (2002) for the process $\widetilde{\mathcal{Z}}_t$ instead of $aW_t + bB^H_t$, where $\widetilde{\mathcal{Z}}_t = aW_t + b\mathcal{Z}_t$, and $\mathcal{Z}$ is a continuous process with vanishing generalized quadratic variation.

Discussion of the Place of Wick Products and Wick–Itô–Skorohod Integral in the Problems of Arbitrage and Replication in the Fractional Black–Scholes Pricing Model

This part of talk is a result of the interesting discussion of the related problems contained in the papers of Sottinen, Valkeila (2003) and Björk, Hult (2005).

The fact of existence of arbitrage in “pure” fractional Brownian model is, to some degree, the consequence of the fact that the mathematical expectation of stochastic integral w.r.t. fBm defined in the path-wise sense is nonzero (and you immediately obtain such integral as a limit of portfolio value created by step buy-and-hold strategies; we discussed this topic in the Subsection 6). Note, however, that arbitrage opportunity constructed by Rogers (1997) does not depend on any particular notion of integration. The same is true for the pre-limit arbitrage of fractional Black-Scholes model considered in Sottinen (2001). Nevertheless, many efforts were made to create the “pure” fractional model which will be “free of arbitrage”, with the help of stochastic integral constructed by Wick products.

Now we present the corresponding list of propositions for alternative definitions of portfolio values and self-financial conditions:

(i) the price of risky asset S is modeled by a geometric fBm and is the solution of the equation

$$dS_t = S_t \diamondsuit dB_t^H, \quad S_0 = s_0, \tag{30}$$

where $H \in (1/2, 1)$ everywhere. In this case

$$S_t = s_0 \exp^{\diamondsuit}(B_t^H) = s_0 \exp\{B_t^H - \frac{1}{2}t^{2H}\} \tag{31}$$

Such approach was developed in Elliott, van der Hoek (2003) and Hu, Øksendal (2003). The portfolio value is defined in the Elliott, van der Hoek. The standard way is $V_t = f_t B_t + g_t S_t$, where f and g are the respective numbers of units of the riskless and the risky asset held in the portfolio. However, in Hu, Øksendal the portfolio value is defined as

$$V_t = f_t B_t + g_t \diamondsuit S_t.$$

The standard Itô-type self-financing condition $dV_t = g_t dS_t$ is replaced by $dV_t = g_t S_t \diamondsuit dB_t^H$ in Elliott, van der Hoek and by $dV_t = g_t \diamondsuit dS_t$ in Hu, Øksendal.

The paper of Björk and Hult claims that the definition of V_t as $V_t = f_t B_t + g_t S_t$ together with $dV_t = g_t S_t \diamondsuit dB_t^H$ (where we put $B_t \equiv 1$) has no economic interpretation as a self-financing condition. Here are the brief arguments. Consider buy-and-hold portfolio. It must satisfy

$$V_t - V_u = g_u(S_t - S_u), \tag{32}$$

from intuitive point of view. However, in our case $V_t - V_u = \int_u^t g_u S_z \diamondsuit dB_z^H$, where the last integral, in general, does not coincide with $g_u \int_u^t S_z \diamondsuit dB_z^H$ and does not coincide with the right-hand side of (32). To be precise with this statement, consider the following example from [2]: let the initial capital $x > 0$, at time $t = 0$ we put our money into the bank account and wait until $t = 1$. Since $B_t \equiv 1$ we receive x at time $t = 1$. At this moment we put our money into the risky asset, i.e., buy x/S_1 shares at the price S_1 and hold this position until $t = 2$. The value of this portfolio at time $t = 2$ is $V_2 = \frac{x}{S_1} S_2$. Evidently, such strategy must be considered as self-similar since nothing was added or subtracted. Nevertheless, $\frac{x}{S_1} S_2 \neq x + \int_0^2 g_u S_u \diamondsuit dB_u^H$ with $g_u = \frac{x}{S_1} \mathrm{1\!I}_{(1,2]}(u)$. Indeed, $E(x + \int_0^2 g_u S_u \diamondsuit dB_u^H)$ exists and equals x, but

$$xE(S_2/S_1) = xE\exp\{B_2^H - B_1^H - 2^{2H-1} + 1/2\} = x\exp\{(1 - 2^{2H})/2\}$$

$$\times E\exp\{B_2^H - B_1^H\} = x\exp\{(1 - 2^{2H})/2\}\exp\{(2-1)^{2H}/2\} = x\exp\{1 - 2^{2\alpha}\},$$

which is not x unless $H \neq 1/2$. There are some other objections concerning this model.

As to model with $dV_t = g_t \diamondsuit dS_t$, simple buy-and-hold strategies will be self-financing in this case. However, the objection in this case is that such definition of portfolio $V_t = f_t dB_t + g_t \diamondsuit dS_t$ is hard to motivate from economic point of view. The reasoning in Björk and Hult are more moral, practical then mathematical: indeed, to calculate the value of portfolio in this case one needs to know Wick calculus and it is hard to instruct the broker how to do it. But there are also some mathematical reasonings against this model, because it can be proved that there exists a portfolio $f = 0$, $g_1 > 0$ such that $g_1 \diamondsuit S_1 < 0$ with positive probability (index 1 stands for the moment of time here). It is sufficient to put $\Omega' = \{\omega \in \Omega | B_1^H(\omega) \in (1/2, 3/2)\}$, $g_1 = S_1 - 1$, where $S_1 = \exp\{B_1^H - 1/2\}$. Then $g_1 > 0$ on Ω', $P(\Omega') > 0$, $g_1 \diamondsuit S_1 = S_1 \diamondsuit S_1 - S_1 = \exp\{2B_1^H - 2\} - \exp\{B_1^H - \frac{1}{2}\} < 0$ on Ω'. In spite of all this criticism, we can say some positive words about Wick (and Skorohod) models with fBm in finances.

First, note that geometric fBm can be written in two forms:

$$S_t^{(1)} = S_0 e^{\mu t + \sigma B_t^H} \quad \text{or} \quad S_t^{(2)} = S_0 e^{\mu t + \sigma B_t^H - \frac{\sigma^2}{2} t^{2H}}. \tag{33}$$

The first form is very simple to understand but the second one is similar to usual geometrical Brownian model $S_t = S_0 e^{\mu t + \sigma B_t - \frac{1}{2}\sigma^2 t}$, because $ES_t^{(2)} = S_0$ for $\mu = 0$.

As it was mentioned in Sottinen, Valkeila, if consider it in the Riemann-Stieltjes sense, the geometric fBm $S_t^{(2)}$ with $\mu = 0$ is the solution of the equation

$$dS_t^{(2)} = S_t^{(2)}(dB_t^H - Ht^{2\alpha}dt), \tag{34}$$

and in the Wick-Skorohod sense $\delta S_t^{(2)} = S_t^{(2)}\delta B_t^H$ or $dS_t^{(2)} = S_t^{(2)} \diamondsuit dB_t^H$, i.e. we obtain the model (30). Nevertheless, due to Riemann-Stieltjes interpretation, we can consider self-financing condition as

$$V_t = V_0 + \int_0^t g_s S_s^{(2)} d(B_s^H - Hs^{2\alpha}ds),$$

and it has a clear economic meaning. Indeed, one can consider the Riemann-Stieltjes integral as an almost sure limit of simple predictable trading strategies.

Now we use the Itô formula for $m = 1$, $S_t := S_t^{(2)}$, $Y_t = \sigma B_t^H + \mu t - \frac{\sigma^2}{2}t^{2H}$, $H \in (1/2, 1)$ and $\widetilde{F}(t, x) = F(t, S_0 e^x)$, take (34) into account and obtain

$$\widetilde{F}(t, Y_t) := F(t, S_t) = F(0, S_0) + \int_0^t \frac{\partial F}{\partial t}(u, S_u)du$$

$$+ \int_0^t \frac{\partial F}{\partial x}(u, S_u)S_u(\mu - H\sigma^2 u^{2\alpha})du + \sigma \int_0^t \frac{\partial F}{\partial x}(u, S_u)d(B_u^H - Hu^{2\alpha}du)$$

$$+H\sigma^2\int_0^t u^{2\alpha}\Big(\frac{\partial^2 F}{\partial x^2}(u,S_u)S_u^2+\frac{\partial F}{\partial x}(u,S_u)S_u\Big)du$$

$$=F(0,S_0)+\int_0^t \frac{\partial F}{\partial t}(u,S_u)du+\mu\int_0^t \frac{\partial F}{\partial x}(u,S_u)S_u du$$

$$+\sigma\int_0^t \frac{\partial F}{\partial x}(u,S_u)d(B_u^H-Hu^{2\alpha}du)+H\sigma^2\int_0^t u^{2\alpha}\frac{\partial^2 F}{\partial x^2}(u,S_u)S_u^2 du.$$

Consider the assumption

$$E\sup_{0\le s\le t}\Big(\frac{\partial F}{\partial x}(s,S_s)S_s\Big)^2+E\sup_{0\le s\le t}\Big(\frac{\partial^2 F}{\partial x^2}(s,S_s)S_s^2\Big)^2<\infty. \tag{35}$$

Let $F(t,S_t):=\widetilde{C}(t,S_t):=C(T-t,S_t)$, where $C(t,x)$ is the price of some European option with $C(T,x)=c(x)$, and S satisfying assumption (35). Then, similarly to (29), we can present $d\widetilde{C}$ in differential form as

$$d\widetilde{C}_t=\sigma\frac{\partial \widetilde{C}}{\partial S}\cdot S(dB_t^H-Ht^{2\alpha}dt)$$

$$+\Big(\mu S\frac{\partial \widetilde{C}}{\partial S}+\frac{\partial \widetilde{C}}{\partial t}+\sigma^2 Ht^{2\alpha}\frac{\partial^2 \widetilde{C}}{\partial S^2}S^2\Big)dt.$$

Now, if the portfolio of value process V consists of one option and an amount of $-\delta$ of underlying assets, then the value $V = \widetilde{C} - \delta \cdot S$, the jump in the value of this portfolio in one-step time equals

$$\begin{aligned} dV_t &= d\widetilde{C}_t - \delta \cdot dS_t \\ &= \sigma \frac{\partial \widetilde{C}}{\partial S} \cdot S_t(dB_t^H - Ht^{2\alpha}dt) - \delta(\sigma S_t(dB_t^H - Ht^{2\alpha}dt)) \\ &\quad + \Big(\mu S_t \frac{\partial \widetilde{C}}{\partial S} + \frac{\partial \widetilde{C}}{\partial t} + \sigma^2 Ht^{2\alpha} \frac{\partial^2 \widetilde{C}}{\partial S^2} S_t^2 - \mu S_t \delta\Big) dt. \end{aligned}$$

If we choose $\delta := \frac{\partial \widetilde{C}}{\partial S}$ to eliminate the stochastic noise, then

$$dV = \Big(\frac{\partial \widetilde{C}}{\partial t} + \sigma^2 Ht^{2\alpha} \frac{\partial^2 \widetilde{C}}{\partial S^2} S^2\Big) dt.$$

The return of an amount V_t invested in bank account equals $rV\,dt$ at time dt. For absence of arbitrage they must be equal, whence we obtain fractional Black–Scholes equation (“Wick” version):

$$\frac{\partial \widetilde{C}}{\partial t} + \sigma^2 Ht^{2\alpha} \frac{\partial^2 \widetilde{C}}{\partial S^2} S^2 + rS \frac{\partial \widetilde{C}}{\partial S} - r\widetilde{C} = 0.$$

We can solve this equation on the segment $[0,T]$ with boundary condition $c(x)=(x-K)^+$, where $K>0$ is strike price, and obtain

$$C(t,S)=\widetilde{C}(T-t,S)=S\Phi\Big(\frac{\ln\frac{S}{K}+r(T-t)+(T^{2H}-t^{2H})\frac{\sigma^2}{2}}{\sigma\sqrt{T^{2H}-t^{2H}}}\Big)$$
$$-Ke^{-r(T-t)}\Phi\Big(\frac{\ln\frac{S}{K}+r(T-t)-(T^{2H}-t^{2H})\frac{\sigma^2}{2}}{\sigma\sqrt{T^{2H}-t^{2H}}}\Big),$$

where $\Phi(\,\cdot\,)$ is a function of standard normal distribution. Note that it coincides with the solution of usual Black-Scholes equation for $H=1/2$.

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