Book Review:
'The Volatility Surface. A Practitioner's Guide'
by Jim Gatheral
(Wiley-Finance, 2006) *

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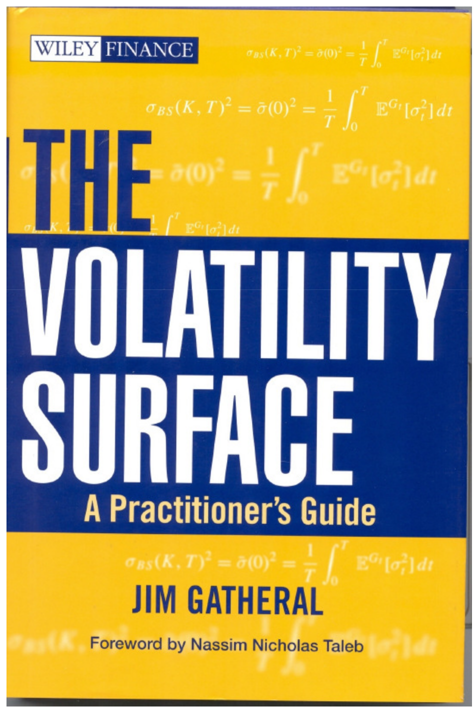
* *Chapter 4: 'The Heston-Nandi Model'*
('Lunch at the Lab' Finance Seminar Reviewing Series)

Book’s Cover


WILEY FINANCE
THE VOLATILITY SURFACE
A Practitioner's Guide
JIM GATHERAL
Foreword by Nassim Nicholas Taleb

Outline of Presentation

1. Intro

2. LV in the Hestin-Nandi Model

3. A Numerical Example

4. Discussion of Results

Chapter 4: The Heston-Nandi Model: Intro

In Chapter 3, we derived approximate formulas for LV and IV in the Heston model:

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sqrt{\nu_t} S_t dZ_t^1 \\ d\nu_t &= -\lambda(\nu_t - \bar{\nu})dt + \eta\sqrt{\nu_t} dZ_t^2, \end{aligned}$$

with $< dZ_1, dZ_2 >= \rho dt$, or ($\mu = 0$)

$$\begin{aligned} dx_t &= -\frac{\nu_t}{2} dt + \sqrt{\nu_t} dZ_t \\ d\nu_t &= -\lambda(\nu_t - \bar{\nu})dt + \rho\eta\sqrt{\nu_t} dZ_t + \sqrt{1-\rho^2}\eta\sqrt{\nu_t} dW_t, \end{aligned}$$

where dW_t and dZ_t are orthogonal and $x_t := \log(S_t/K)$.

Chapter 4: The Heston-Nandi Model: Intro

In this Chapter 4, we compute LV and IV for a particular choice of Heston parameters for which equation for LV

$$u_T \approx \widehat{v}'_T + \rho\eta \frac{x_T}{\widehat{w}_T} \int_0^T \widehat{v}_s e^{-\lambda'(T-s)} ds,$$

where $u_t := E[v_t | x_T]$, $\quad \widehat{w}_t := \int_0^t \widehat{v}_s ds, \lambda' = \lambda - \rho\eta/2, \bar{v}' = \bar{v}\lambda/\lambda'$, $\quad \widehat{v}'_s := (v - \bar{v}')e^{-\lambda' s} + \bar{v}'$, $\quad \widehat{v}_s = (v_0 - \bar{v})e^{-\lambda s} + \bar{v}$

gives a very good approximation to the true LV.

Chapter 4: The Heston-Nandi Model: Intro

This provides us with a specific set of Heston parameters and LV that we use in subsequent chapters to study the impact of modeling assumptions on the valuation of various kinds of options, confident that both LV and Heston models generate the same European option prices.

Chapter 4: The Heston-Nandi Model: LV in the Heston-Nandi Model

Following the derivation in Chapter 3, we see that if $\rho = -1$, the formula presented for LV should be pretty good. In this case, because $\rho = -1$, the Heston process is only one-factor and the SDE can be written as

$$\begin{aligned} dx &= -\frac{\nu}{2} dt + \sqrt{v} dZ \\ dv &= -\lambda(v - \bar{v}) dt - \eta \sqrt{v} dZ. \end{aligned}$$

Chapter 4: The Heston-Nandi Model: LV in the Heston-Nandi Model

The choice $\rho = -1$ was originally studied by Heston and Nandi (1998) as the preference-free continuous time limit of a discrete GARCH option pricing model previously introduced by them. Their model was preference-free because there is only one source of randomness. So all volatility risk can be eliminated by appropriately delta hedging with stock; there is no volatility risk premium in this case.

Chapter 4: The Heston-Nandi Model: LV in the Heston-Nandi Model

Although the Heston model is only one factor in this special case, it is certainly not Markov in the stock price. That's because the instantaneous volatility is a deterministic function of the entire history of the stock price and in general, computing an expectation under the risk-neutral measure requires knowledge of the volatility.

Chapter 4: The Heston-Nandi Model: LV in the Heston-Nandi Model

To see this clearly, we can rewrite the SDE for v as

$$dv = -\lambda'(v - \bar{v}')dt - \eta dx,$$

with $\lambda' = \lambda + \eta/2, \bar{v}' = \bar{v}\lambda/\lambda'$. Note also, that although zero instantaneous variance may be attainable depending on the value of the parameters, it can never be negative. In particular, LV can never be negative.

Chapter 4: The Heston-Nandi Model: LV in the Heston-Nandi Model

LV in this special case (from equation (3.11): $\sigma_{BS}(K,T)^2 \approx \frac{1}{T}\int_0^T v_L(\tilde{x}_t)dt$) is given by

$$
\begin{aligned}
v_L(x_T,T) &= \widehat{v}'_T - \eta \frac{x_T}{w_T}\int_0^T \widehat{v}_s e^{-\lambda'(T-s)}ds \\
&= (v-\bar{v}')e^{\lambda' T} + \bar{v}' - \eta x_T[\tfrac{1-e^{-\lambda' T}}{\lambda' T}].
\end{aligned}
$$

The whole expression must be bounded below by zero-all stock prices above the critical stock price at which the LV reaches zero are unattainable.

Chapter 4: The Heston-Nandi Model: A Numerical Example

In order to assess the accuracy of the approximate LV formula in Chapter 3

$$u_T \approx \widehat{v}_T' + \rho\eta \frac{x_T}{\widehat{w}_T} \int_0^T \widehat{v}_s e^{-\lambda'(T-s)} ds,$$

Chapter 4: The Heston-Nandi Model: A Numerical Example

while also exploring some properties of the Heston-Nandi model, we fix Heston parameters as follows:

$$v = 0.04, \quad \bar{v} = 0.04, \quad \lambda = 10, \quad \eta = 1, \quad \rho = -1.$$

Author uses these parameters repeatedly in Heston computations throughout the rest of the book.

Chapter 4: The Heston-Nandi Model: A Numerical Example: The Heston-Nandi Density

To get the Heston-Nandi probability density $p(k,T)$ for a given expiration T, we invert the Heston characteristic function $\phi_T(u)$,

$$p(k,T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi_T(u) e^{-iuk} du$$

Chapter 4: The Heston-Nandi Model: A Numerical Example: The Heston-Nandi Density

with

$$\phi_T(u) = \exp[C(u,\tau)\bar{v} + D(u,\tau)v],$$

where C and D are defined in Chapter 2:

$$\begin{aligned} C(u,\tau) &= \lambda\{r_-\tau - \tfrac{2}{\eta^2}\log(\tfrac{1-ge^{-d\tau}}{1-g})\} \\ D(u,\tau) &= r_-\tfrac{1-e^{-d\tau}}{1-ge^{-d\tau}}, \end{aligned}$$

where $g := r_-/r_+, r_\pm = \frac{\beta\pm\sqrt{\beta^2-4\alpha\gamma}}{2\gamma}, d = \sqrt{\beta^2 - 4\alpha\gamma}$.

Chapter 4: The Heston-Nandi Model: A Numerical Example

Computing $p(k,T)$ numerically with $T = 0.1$ years and the above parameters generates the plot shown in Figure 4.1. It's easy to see from this plot that stock prices above some critical stock price are unattainable in the H-N model; there is a critical strike price above which call options have zero value. This observation alone makes the H-N model look rather unrealistic.

The Probability Density for the Heston-Nandi Model

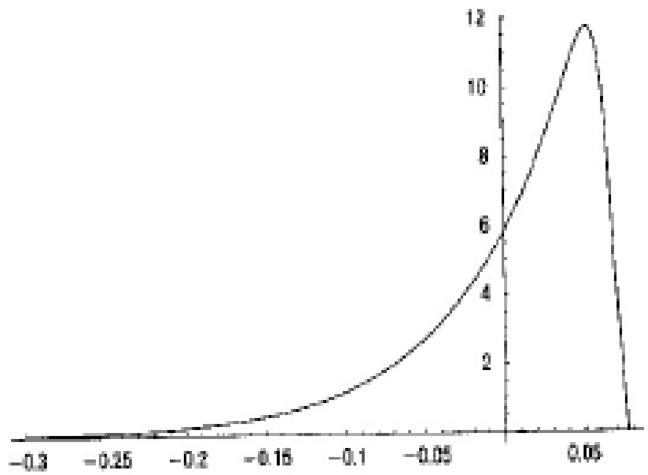


FIGURE 4.1 The probability density for the Heston-Nandi model with our parameters and expiration $T = 0.1$.

Chapter 4: The Heston-Nandi Model: A Numerical Example: Computation of LV

From Chapter 3, LV is obtained from the Dupire equation,

$$\frac{\partial C}{\partial T} = \frac{\sigma^2 K^2}{2}\frac{\partial^2 C}{\partial K^2} + (r_t - D_t)(C - K\frac{\partial C}{\partial K})$$

as the ratio of a calendar spread (time derivative) to a butterfly (probability density).

Chapter 4: The Heston-Nandi Model: A Numerical Example: Computation of LV

We have already computed the density $p(k, T)$ so it only remains for us to compute the calendar spread. This we may do by differentiating the Heston call value wrt time to expiration τ. From equation (2.5) for C :

$$C(x, \nu, \tau) = K[e^x P_1(x, \nu, \tau) - P_0(x, \nu, \tau)],$$

we get

Chapter 4: The Heston-Nandi Model: A Numerical Example: Computation of LV

$$\partial_\tau C(x,v,\tau) = K[e^x \partial_\tau P_1(x,v,\tau) - \partial_\tau P_0(x,v,\tau)]$$

with the $P_j(\cdot)$ are from equation (2.13):

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} du Re[\frac{\exp(C_j\bar{v}) + D_j v + iux}{iu}].$$

Chapter 4: The Heston-Nandi Model: A Numerical Example

Inverting equation for C then leads to

$$v_L(x_t, \tau) = 2\frac{\partial_\tau c(x_t, \tau)}{p(k, T)}$$

with

$$c(x, \tau) = \frac{C(x, v, \tau)}{K}.$$

Figure 4.2 shows the results of computing LV in the H-N model using the exact (but possibly numerically inaccurate) formula for v_L and the approximate (but numerically accurate) last formula.

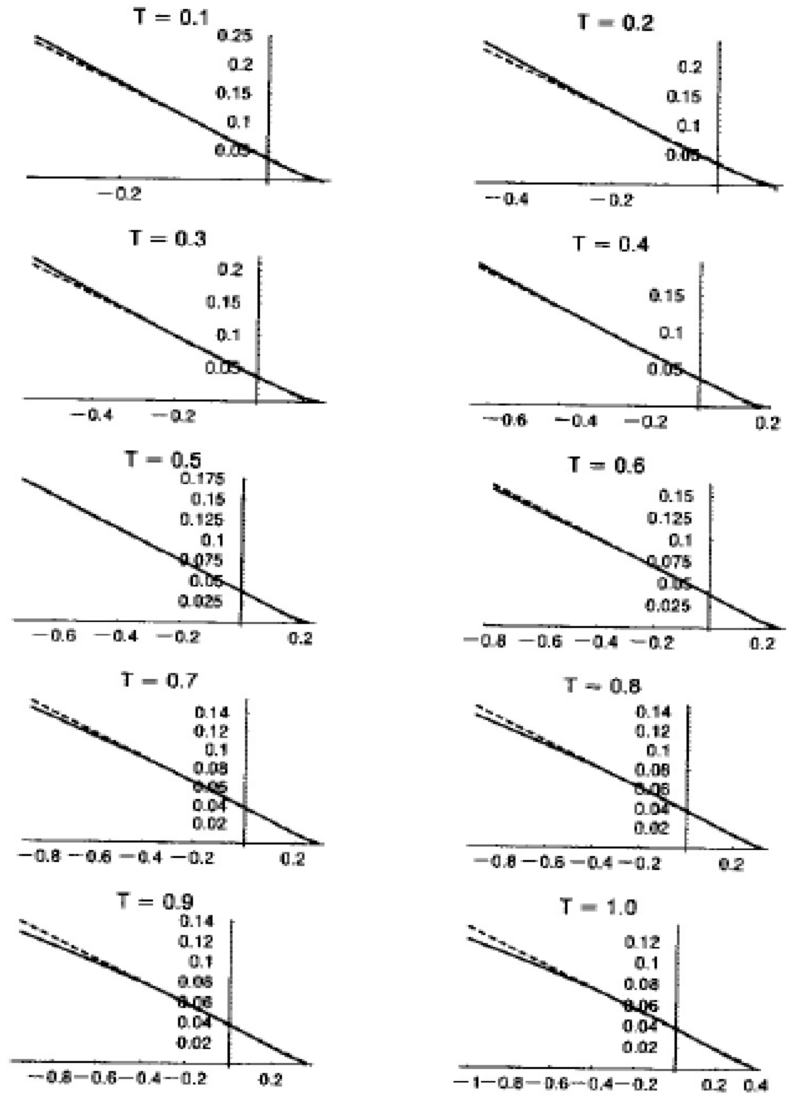


FIGURE 4.2 Comparison of approximate formulas with direct numerical computation of Heston local variance. For each expiration T, the solid line is the numerical computation and the dashed line is the approximate formula.

Chapter 4: The Heston-Nandi Model: A Numerical Example: Computation of IV

The reader may or may not be convinced by the close agreement between the approximate and exact LV plotted in Figure 4.2. For one thing, it's unclear whether errors at low strikes are due to the inaccuracy of the approximation or numerical inaccuracy in the computation of the exact LV.

Chapter 4: The Heston-Nandi Model: A Numerical Example: Computation of IV

The proof of the pudding is in the eating: All we need is for the prices of European options to agree. We therefore compute European option prices using the Heston formula (2.13),

$$P_j(x, \nu, \tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} du Re[\frac{\exp(C_j\bar{v}) + D_j v + iux}{iu}].$$

(we note, that calculations of delta and vega (derivatives wrt x and ν) are strightforward because the functions $C(u, \tau)$ and $D(u, \tau)$ are independent of x and ν)

Chapter 4: The Heston-Nandi Model: A Numerical Example: Computation of IV

and again by solving the LV valuation equation numerically (see Tavella and Randall (2000)) with LV given by equation for $v_L(x_T, T)$.

Chapter 4: The Heston-Nandi Model: A Numerical Example: Computation of IV

Explicitly, the numerical PDE to be solved for an option with strike K and expiratuion T is

$$\frac{\partial V}{\partial t} + \frac{1}{2} v(S,t) \frac{\partial^2 V}{\partial S^2} = 0$$

subject to the boundary condition $V(S_T, T) = (S_T - K)^+$, where from equation for v_L :

$$\begin{aligned} v_L(x_T, T) &= \widehat{v}'_T - \eta \tfrac{x_T}{w_T} \textstyle\int_0^T \widehat{v}_s e^{-\lambda'(T-s)} ds \\ &= (v - \bar{v}') e^{\lambda' T} + \bar{v}' - \eta x_T [\tfrac{1-e^{-\lambda' T}}{\lambda' T}]. \end{aligned}$$

Chapter 4: The Heston-Nandi Model: A Numerical Example: Computation of IV

$$v(S,t) = (v - \bar{v}')e^{-\lambda' T} + \bar{v}' - \eta \log(S/S_0)[\frac{1 - e^{-\lambda'}T}{\lambda' T}].$$

In Figure 4.3, we see again that agreement is very close.

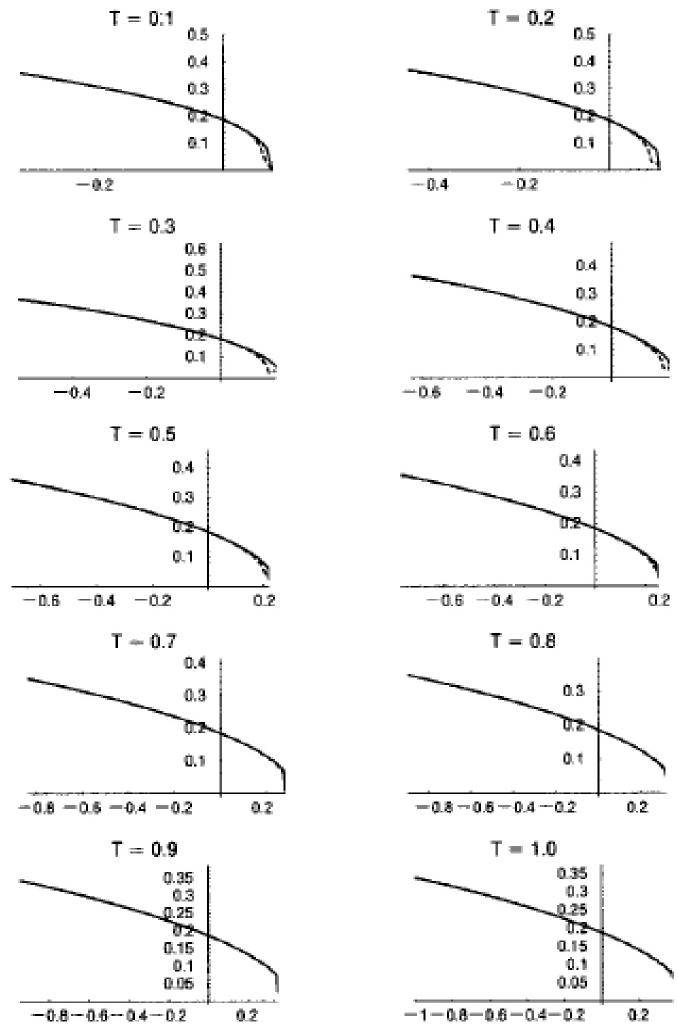


FIGURE 4.3 Comparison of European implied volatilities from application of the Heston formula (2.13) and from a numerical PDE computation using the local volatilities given by the approximate formula (4.1). For each expiration T, the solid line is the numerical computation and the dashed line is the approximate formula.

Chapter 4: The Heston-Nandi Model: Discussion of the Results

From the results of our computation, we can see that the LV model and the SV model price European options almost identically. Thus we have created a toy set of market parameters that will allow us to compare the effects of SV and LV assumptions on the valuation of various claims, confident that European options are almmost identically priced under both sets of assumptions.

Chapter 4: The Heston-Nandi Model: Discussion of the Results

We note too that both the Heston model and its LV equivalent are single-factor, depending only on stock prices. However, the two models are clearly not equivalent: in the LV model, volatilites are known in advance and in the SV case, volatilites are uncertain.

Chapter 4: The Heston-Nandi Model: Discussion of the Results

The consequences of this fundamental difference between the two models will become clear as we proceed to value various exotic options in succeeding chapters.

In other words, to value an option, it's not enough just to fit all the European option prices, we also need to assume some specific dynamics for the underlying.

Conclusion

1. LV in the Hestin-Nandi Model

2. A Numerical Example

3. Discussion of Results

The End of Chapter 4

Thank You for Your Time and Attention!

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