Volatility Surface Asymptotics

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Abstract

In this talk, we are concerned with the shape of the volatility surface for very generic models of stochastic volatility with jumps type. We present many results about the asymptotics of the BS implied volatility.

1 Introduction

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The variation of implied volatilities (IVs) accross strike and term to maturity which is widely referred to as the volatility surface can be sustantial. Since the 1987 market crash, volatility suraface for global indices have been characterized by volatility skew. Conventional explanations for the volatility skew include the lerage effect (stocks tend to be more volatile at lower prices than higher), the risk of default, and suppy and demand, the fact that volatility moves and spot moves are anticorrelated. It is reasonable to expect skew to play a role in relating the magnitudes of these changes.

2 Short Expirations

We consider the stock price model described by the following syatem of SDEs:
$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dZ_1$

$dv_t = \alpha(S_t, v_t, t)dt + \eta\beta(S_t, v_t, t)\sqrt{v_t}dZ_2$ (7.1)
where S_t stands for the stock price at time t,
μ_t is the (deterministic) instantaneous drift of stock price returns,
$v_t = \sigma_t^2$ is the instantaneous variance of stock price returns,
η is the volatility of volatility,
Z_1 and Z_2 are two Wiener processes correlated with a correlation coefficient ρ. (We can express this by writing: $dZ_2 = \rho dZ_1 + \phi dZ$, where Z is independent of Z_1).
We wish to rewrite (7.1)under the risk neutral measure(determined by market preferences)and in terms of the log-moneyness $x = \log(F(t,T)/K)$, where K is the strike price, T the expiration date, $F(t,T) = S_t \exp(\int_t^T \mu_s ds)$ is the forward price of the stock at time $t \in [0,T]$.
Setting $f(S_t) = \log(S_t)$,where $f(x) = \log(x)$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$,
we have by the Ito formula:

$$\begin{aligned} d\log(S_t) &= df(S_t) \\ &= f'(S_t)dS_t + \frac{1}{2}f''(x)d[S]_t \\ &= \frac{1}{S_t}dS_t - \frac{1}{2(S_t)^2}v_t(S_t)^2dt \\ &= \mu_t dt + \sqrt{v_t}dZ_1 - \frac{1}{2}v_t dt \end{aligned}$$

so that we have:

$$\begin{aligned} dx &= d\log(S_t) - \mu_t dt \\ &= \mu_t dt + \sqrt{v_t}dZ_1 - \frac{1}{2}v_t dt - \mu_t dt \\ &= -\frac{1}{2}v_t dt + \sqrt{v_t}dZ_1 \end{aligned}$$

Assuming that α and β do not depend on S or t, the second equation of (7.1) can be written as:

$$\begin{aligned} dv &= \alpha(v)dt + \eta\beta(v)\sqrt{v}dZ_2 \\ &= \alpha(v)dt + \eta\beta(v)\sqrt{v}(\rho dZ_1 + \phi dZ) \\ &= \alpha(v)dt + \rho\eta\beta(v)(dx + \frac{1}{2}vdt) + \eta\phi\beta(v)\sqrt{v}dZ \end{aligned}$$

so that $E[v + dv|x] = v + \alpha(v)dt + \rho\eta\beta(v)(dx + \frac{1}{2}vdt)$.
Therefore, for small times to expiration (relative to the variations of $\alpha(v)$ and $\beta(v)$), we have:
$v_{loc}(x,t) = E[v_t|x_t = x]$ hence $v_{loc}(x,t) \approx v_0 + [\alpha(v_0) + \frac{1}{2}\rho\eta v_0\beta(v_0)]t + \rho\eta\beta(v_0)x$ (7.2)
Here, the local variance skew $\rho\eta\beta(v_0)$ agrees with that derived by Lee(2001).
To extend the result to the BS implied variance skew, we state and prove the following lemma.

2.1 Lemma

The local variance skew is twice as steep as the BS implied variance skew.

Proof.
Recall that the Black-Scholes implied variance of an option with strike K is given approximately by the integral from valuation date ($t = 0$) to the expiration date ($t = T$) of the local variances along the path $\widetilde{x}_t$ that maximizes the Brownian bridge density $q(x_t, t; x_T, T)$(See page 30, Chapter 3).

$\sigma_{BS}(K,T)^2 \approx \frac{1}{T}\int_0^T v_{loc}(\widetilde{x}_t, t)dt$

$\approx const. + \frac{1}{T}\int_0^T \rho\eta\beta(v_0)\widetilde{x}_t dt$

$\approx const. + \frac{1}{T}\int_0^T \rho\eta\beta(v_0)x_T\frac{t}{T}dt$

$= const. + \frac{1}{2}\rho\eta\beta(v_0)x_T$

Thus, for short times to expiration, the BS implied variance skew is given by:
$\frac{\partial\sigma_{BS}(K,T)^2}{\partial x_T} = \frac{1}{2}\rho\eta\beta(v_0)$ (7.3)
Note that equation (7.3) is consistent with the short-dated volatility skew behavior seen for the Heston model on [1], Page 35, Chapter 3, since under this model, $\beta(v)$ is the constant 1.

3 The Medvedev-Scaillet Result

In the setting where stochastic volatility and jumps drive the dynamics of stock returns, Medvedev and Scaillet (2004) derived a short term asymptotics

expansion of implied volatility for fixed moneyness degree (or normalized log-strike) z defined as:
$z = \left(\frac{k}{\sigma_{BS}(x,\tau)\sqrt{\tau}}\right)$, where $k = \log(K/S)$.
When the underlying process is a pure diffusion of the form:
$dS_t = \sigma_t S_t dZ_1$
$d\sigma_t = a(\sigma_t)dt + b(\sigma_t)dZ_2$ (7.4), they proved that the implied volatility I has the following short term asymptotics:
$I(z,\tau,\sigma) = \sigma + I_1(z,\sigma)\sqrt{\tau} + I_2(z,\sigma)\tau + O(\tau\sqrt{\tau})$,
where $I_1(z,\sigma) = \frac{1}{2}b\rho z$
$I_2(z,\sigma) = \frac{1}{6}[\frac{b^2(1-\rho^2)}{\sigma} + bb'\rho^2]z^2 + \frac{1}{2}a + \frac{b\rho\sigma}{4} + \frac{b^2\rho^2}{24\sigma} + \frac{b^2}{12\sigma} - \frac{bb'\rho^2}{6}$, (7.5)
with $a = a(\sigma)$, $b = b(\sigma)$, $b' = \frac{\partial b(\sigma)}{\partial\sigma}$.
It is worth observing that the limit of implied volatility I as the moneyness degree z tends to zero and the time to expiration τ tends to zero is the instantaneous volatility σ.
Now, plugging $z = \frac{k}{I(z,\tau,\sigma)\sqrt{\tau}}$ into (7.5) and taking the limit as τ tends to zero gives :
$\frac{\partial I}{\partial k}|_{k=0} = \frac{b(\sigma)\rho}{2\sigma}$(7.6), which proves our earlier result (7.3). It is useful to notice that the short-dated volatility skew is not explicitly time-dependent; its depends only on the form of the SDE for volatility. In contrast, for local volatlity models, short-dated skews decay rapidly as time avances. So, even if a stochastic volatility model and a local volatility model price all European options identically today, the volatility surface dynamics of the two models are quite different.

4 The SABR model

The Stochastic Alpha Beta Rho (SABR) model (introduced by Hagan, Kumar, Lesniewsi and Woodward (2002)) is a model of stochastic volatility with dynamics:
$dS_t = \sigma_t S_t^{\beta} dZ_1$
$d\sigma_t = \chi\sigma_t dZ_2$,
where Z_1 and Z_2 are two correlated Wiener processes with correlation coefficient $-1 < \rho < 1$,
and the parameters β and χ satisfy the conditions: $0 \leq \beta \leq 1$ and $\chi \geq 0$.
The prices of European options in this model are given by the Black-Scholes model. From these prices, closed-form algebraic formulas for the implied

volatility are derived. These fomulas show that the SABR model captures the correct dynamics of the implied volatility smile for short expirations. It is useful to notice that this model is only good for short expirations since under it, we do not have mean reversion of volatility.
The case $\beta = 0$ produces the stochastic normal model, the case $\beta = \frac{1}{2}$ produces the stochastic CIR model and the case $\beta = 1$ the stochastic lognormal model.
In the case $\beta = 1$, the SABR implied volatility reads:
$\sigma_{BS}(k, \tau) = (y\sigma_0/f(y))[1 + [\frac{\rho\chi\sigma_0}{4} + \frac{(2-3\rho^2)\chi^2}{24}]\tau + O(\tau\sqrt{\tau})]$(7.7)
with $y = -\frac{k\chi}{\sigma_0}$
and $f(y) = \log[\frac{\sqrt{1-2\rho y+y^2}+y-\rho}{1-\rho}]$
. Expanding to 2nd order in y and first order in τ, with $y \sim \sqrt{\tau}$, we get:

$$\begin{aligned}\sigma_{BS}(k, \tau) = \sigma_0 \Big\{ 1 - \tfrac{\rho y}{2} + \tfrac{(2-3\rho^2)y^2}{12} \\ + \left[\tfrac{1}{4}\rho\chi\sigma_0 + \tfrac{1}{24}(2 - 3\rho^2)\chi^2\right]\tau + O(\tau\sqrt{\tau})\Big\}\end{aligned}$$

Substituting $a = a(\sigma) = 0$ and $b = b(\sigma) = \chi\sigma_0$ into (7.5) gives:
$I_1(z, \sigma_0) = \frac{1}{2}\rho\chi\sigma_0 z$

$$I_2(z, \sigma_0) = \frac{\chi^2\sigma_0}{z}^2 + \frac{\rho\chi\sigma_0^2}{4} + \frac{\rho^2\chi^2\sigma_0}{24} + \frac{\chi^2\sigma_0}{12} - \frac{\rho^2\chi^2\sigma_0}{6}$$

Noting that

$$\chi z\sqrt{\tau} = \chi\frac{k}{\sigma_{BS}} = -y\frac{\sigma_0}{\sigma_{BS}} = -y(1 + \frac{\rho y}{2}) + O(y^3)$$

we obtain

$$\begin{aligned}\sigma_{BS}(k, \tau) = \sigma_0 + I_1(z, \sigma_0)\sqrt{\tau} + I_2(z, \sigma_0)\tau + O(\tau\sqrt{\tau}) \\ = \sigma_0 \Big\{ 1 - \tfrac{\rho y}{2} + \tfrac{(2-3\rho^2)y^2}{12} + \left[\tfrac{1}{4}\rho\chi\sigma_0 + \tfrac{1}{24}(2 - 3\rho^2)\chi^2\right]\tau + O(\tau\sqrt{\tau})\Big\}\end{aligned}$$

Clearly, the Medvedev-Scaillet result (7.5)agrees with the SABR implied volatility formula (7.7) for small times to expiration.
It should be added that the SABR formula gives:
$\frac{\partial\sigma_{BS}}{\partial k}|_{k=0} = \frac{\rho\chi}{2}$, which is a special case of the result (7.3) with $\beta(v) = 1$ and $\eta = 2\chi$.

5 Including Jumps

Medvedev and Scaillet noted that an introduction of jumps is the natural way to make the model more realistic. However, jumps in volatility only affect the short-term volatility skews.
We consider a stochastic volatility with jumps model. Under the risk neutral measure, the joint dynamics of the stock price and its volatility is:
$\frac{dS_t}{S_t} = \sigma_t dZ_1 + J(\sigma_t)dq_t$
$d\sigma_t = a(\sigma_t)dt + b(\sigma_t)dZ_2$,
where q is a Poisson counting process with intensity $\lambda_J(\sigma_t)$(that may depend on the volatility in a deterministic way but the occurence of jumps is independent of the Wiener process Z_1),
$J(\sigma_t)$ is a $(-1, \infty)$-valued random variable with density f sampled at each jump.
The jump compensator (or expected jump size) is then:
$\mu_J = \lambda_J E(\Delta J)$
$\quad = \lambda_J \int_{-1}^{\infty} x f(x) dx$

Under this mixed jump-diffusion model, the short term asymptotics of implied volatility is of the form:
$I(z, \tau, \sigma)$
$\quad = \sigma + \overline{I_1(z, \sigma)}\sqrt{\tau} + \overline{I_2(z, \sigma)}\tau + O(\tau\sqrt{\tau})$
We refer to [1] for the details of the functions $\overline{I_1(z, \sigma)}$ and $\overline{I_2(z, \sigma)}$.

6 Corollaries

In a jump diffusion model with deterministic volatility, the limit of the implied volatility skew as the time to expiration τ tends to zero is given by:
$\frac{\partial I}{\partial k}|_{k=0} \to -\frac{\mu_J}{\sigma}$
, which is consistent with the derivation on [1], Page 62, Chapter 5.

Under the stochastic volatility model with jumps, the limit of the implied volatility skew as τ tends to zero is given by:
$\frac{\partial I}{\partial k}|_{k=0} \to \frac{\rho(\sigma)}{2\sigma} - \frac{\mu_J}{\sigma}$. This agrees with the observation that stochastic volatility and jump effects on the at-the-money variance skew are roughly additive (See [1], Chapter 5). In fact, we have:
$\frac{\partial v_{BS}}{\partial k}|_{k=0} \to \rho b(\sigma) - 2\mu_J$ as τ tends to zero.

7 Long Expirations

Fouque, Papanicolau and Sircar(1999) and Fouque, Papantonicolau and Sircar(2000) analyzed a model in which stock prices are conditionally normal and the log-volatility is an Ornstein-Uhlenbeck (OU) process. That is,
$dx = -\frac{\sigma^2}{2}dt + \sigma dZ_1$
$d\log(\sigma) = -\lambda[\log(\sigma) - \overline{\log(\sigma)}]dt + \xi dZ_2,$
where $x = \log(F(t,T)/K)$.
They deduced for large λT the slope of the BS implied volatility skew:
$\frac{\partial \sigma_{BS}(x,T)}{\partial x_T} \approx \frac{\rho\xi}{\lambda T}$ (7.10).

Now, considering random terms only, $dv \sim 2\sigma d\sigma$ and in the log-OU model, $d\sigma \sim \xi\sigma dZ_2$

so that $dv \sim 2\xi v dZ_2$.

Thus $\beta(v)$ as defined in (7.1) satisfies $\eta\beta(v) = 2\xi\sqrt{v}$ and from equation (7.10),

we deduce:
$\frac{\partial \sigma^2_{BS}}{\partial x_T} \approx \frac{2\rho\xi\sqrt{v}}{\lambda T}$

Hence $\frac{\partial \sigma_{BS}}{\partial x_T} \approx \frac{\rho\eta\beta(v)}{\lambda T}$.

Furthermore, in the particular example of the Heston model, the BS implied variance skew has the same form for both long expirations as in here(See (3.19), Page 35) and short expirations, which could provide a way of interpolating between these two times (See (7.11),[1], Page 96).

8 Small volatility of volatility

Using an expansion of a call option price in terms of the volatility of volatility η assumed small in any model of the form (7.1), Lewis (2000) obtained the following expansion for the implied variance.
$v_{BS}(x,\tau) = \beta_0(v,\tau) + \beta_1(v,\tau)x + \beta_2(v,\tau)x^2 + O(\eta^3)$
where $x = \log(F(t,T)/K)$. For the expressions of the functions $\beta_j(.)$, we refer to [3].
Considering the dynamics: $dv = -\lambda(v - \overline{v})dt + \eta v^{\Phi}dZ$, the at-the-money skew reads:

$\frac{\partial v_{BS}}{\partial k}|_{k=0}$
$= \frac{\rho\eta v^{\Phi-1/2}}{\lambda t}\left\{1 - \frac{1-\exp(-\lambda t)}{\lambda t}\right\},$

which agrees with (7.1) to a volatility of volatility factor.

9 Extreme Strikes

In a working paper, Roger Lee (2004) established that (for options on a non-negative underlying random variable with arbitrary distribution) in the absence of arbitrage, at any maturity T, the (large-strike tail of) the BS implied volatility skew is bounded above by a function linear in the log-moneyness $x = \log(F/K)$ as $|x|$ tends to ∞. Moreover, he showed that there is a one-to-one relationship between the large-strike tail slope of the BS implied volatility and the maximal number of finite moments of the underlying process.
Theorem: Moment Formula, part 1 Let $\overline{p} = sup\left\{p : ES_T^{1+p} < \infty\right\}$ and $\overline{\alpha} = \lim sup_{x\to\infty} \frac{T\sigma_{BS}(x,T)^2}{|x|}$

Then $\overline{\alpha} \in [0, 2]$, $\overline{p} = \frac{1}{2}\left\{\frac{1}{\sqrt{\overline{\alpha}}} - \frac{\sqrt{\overline{\alpha}}}{2}\right\}$ Equivalently, $\overline{\alpha} = g(\overline{p})$,
where $g(x) = 2 - 4[\sqrt{x^2 + x} - x]$.

Theorem: Moment Formula, part 2

Let $\overline{q} = sup\left\{q : ES_T^{-q} < \infty\right\}$ and $\overline{\beta} = \lim sup_{x\to-\infty} \frac{T\sigma_{BS}(x,T)^2}{|x|}$

Then $\overline{\beta} \in [0, 2]$, $\overline{q} = \frac{1}{2}\left\{\frac{1}{\sqrt{\overline{\beta}}} - \frac{\sqrt{\overline{\beta}}}{2}\right\}$ Equivalently, $\overline{\beta} = g(\overline{q})$,
where $g(x) = 2 - 4[\sqrt{x^2 + x} - x]$.
Benaim and Fritz (2006) established that Roger Leee's upper bound may be replaced by a limit, provided that $\log[1 - F(x)]$ and $\log[F(-x)$ satisfy some conditions (that are satisfied in most models of practical interest).

10 Concluding Remarks

Clearly, the specific choice of the model has very little effect on the general shape of the volatility surface. Hence, it is practically impossible to deduce anything about the specific form of the volatility dynamics from a single observation of the volatility surface.

11 References

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[4] R. Lee. Implied volatility: Statics,Dynamics and Probabilistic Interpretation. Recent Advances in Applied Probability, Springer, (2004).