Book Review:

'The Volatility Surface. A Practitioner's Guide'

by Jim Gatheral

(Wiley-Finance, 2006) *

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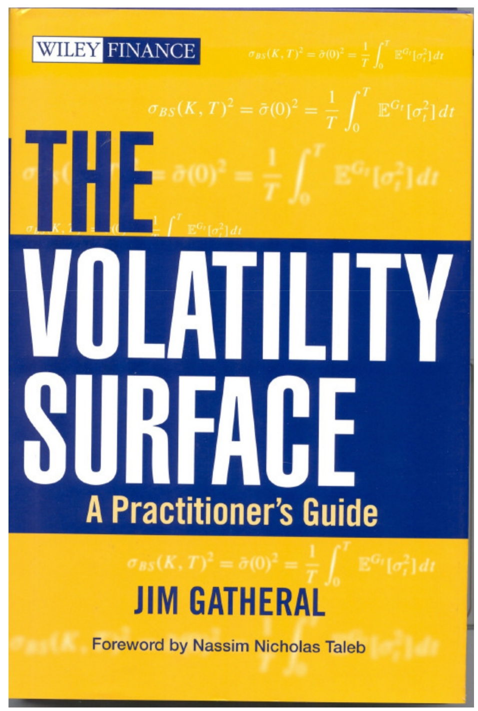
'Lunch at the Lab' Talk

January 20, 2010

* *Chapter 8: 'Dynamics of the Volatility Surface'*
('Lunch at the Lab' Finance Seminar Reviewing Series)

Book’s Cover


WILEY FINANCE
THE VOLATILITY SURFACE
A Practitioner's Guide
JIM GATHERAL
Foreword by Nassim Nicholas Taleb

Outline of Presentation

1. Intro

2. Dynamics of the Volatility Skew under SV

3. Dynamics of the Volatility Skew under LV

4. Stochastic IV Models

5. Digital Options and Digital Cliquets

Asymptotics in Summary (Chapter 7: Volatility Surface (VS) Asymptotics)

It is quite clear from the results presented in Chapter 7 that the general shape of the VS does not depend very much on the specific choice of model.

Any SV with jump model should generate a similar shape of VS with appropriate numerical choices of the parameters.

Chapter 8: Dynamics of the Volatility Surface: Intro

In Chapter 7, we saw that all SV models have essencially the same implications for the shape of the volatility surface. At first it might seem that it would be hard to differentiate between models. That would certainly be the case if we were to confine our attention to the shape of the volatility surface today.

However, if instead we were to study the dynamics of the volatility skew-in particular, how the observed volatility skew depends on the overall of volatility, we would be able to differentiate between models.

Chapter 8: Dynamics of the Volatility Surface: Dynamics of the Volatility Skew under SV

Empirical studies of the dynamics of the volatility skew show that $\frac{\partial \sigma(k,t)}{\partial k}$ is approximately independent of volatility level over time. Translating this into a statement about the implied variance skew, we get

$$\frac{\partial \sigma_{BS}(k,t)^2}{\partial k} = 2\sigma_{BS}(k,t)\frac{\partial \sigma_{BS}(k,t)}{\partial k} \approx \sqrt{v(k,t)}.$$

Chapter 8: Dynamics of the Volatility Surface: Dynamics of the Volatility Skew under SV

Comparing this with equation (7.11) (here, $\lambda' = \lambda - \frac{1}{2}\rho\eta\beta(v)$)

$$\frac{\partial \sigma_{BS}(x,t)^2}{\partial x} \approx \frac{\rho\eta\beta(v)}{\lambda' T}[1 - \frac{(1 - e^{-\lambda' T})}{\lambda' T}]$$

we see that this in turn implies that $\beta(v) \approx \sqrt{v}$. Referring back to the definition of $\beta(v)$ in (7.1),

$$dv = \alpha(v)dt + \eta\sqrt{v}\beta(v)dZ_2,$$

we conclude that v is approximately lognormal in contrast to the square root process assumed by Heston. This makes intuitive sense given that we would expect volatility to be more volatile if the volatility level is high than if the volatility level itself is low.

Chapter 8: Dynamics of the Volatility Surface: Dynamics of the Volatility Skew under SV

Does it matter whether we model variance as a square root process or as lognormal? It does.

Is any SV model better than none at all? Yes, because, whereas having the wrong SV model will cause the hedger to generate a payoff corresponding to a skew that may be off by a factor of 1.5 if volatility doubles, having only a LV model will cause the hedger to generate a payoff that corresponds to almost no forward skew at all. We now show this.

Chapter 8: Dynamics of the Volatility Surface: Dynamics of the Volatility Skew under LV

Empirically, the slope of the volatility skew decreases with time to expiration. From the above, in the case of mean-reverting SV, the term structure of the B-S implied variance skew will look something like equation (7.11):

$$\frac{\partial \sigma_{BS}(x,t)^2}{\partial x} \approx \frac{\rho \eta \beta(v)}{\lambda' T}[1 - \frac{(1-e^{-\lambda' T})}{\lambda' T}].$$

Recall from Chapter 1 the formula (1.10) for LV in terms of IV ($w = \sigma^2(K,T;S_0)T$)

$$v_L = \frac{\frac{\partial w}{\partial T}}{1 - \frac{k}{w}\frac{\partial w}{\partial k} + \frac{1}{4}(-\frac{1}{4} - \frac{1}{w} + \frac{k^2}{w^2})(\frac{\partial w}{\partial k})^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}}$$

Chapter 8: Dynamics of the Volatility Surface: Dynamics of the Volatility Skew under LV

Differentiating wrt x and considering only the leading term in $\frac{\partial w}{\partial k}$ (which is small for large T), we find

$$\frac{\partial v_L}{\partial k} \approx \frac{\partial}{\partial T}\frac{\partial w}{\partial k} + \frac{1}{w}\frac{\partial w}{\partial T}\frac{\partial w}{\partial k}.$$

That is, the LV skew $\frac{\partial v_L}{\partial k}$ decays with the B-S implied total variance skew $\frac{\partial w}{\partial k}$.

Chapter 8: Dynamics of the Volatility Surface: Dynamics of the Volatility Skew under LV

To get the forward volatility surface from the LV surface in a LV model, we integrate over the LVs from the (forward) valuation date to the expiration of the option along the most probable path joining the current stock price to the strike price using the trick presented in Chapter 3.

Chapter 8: Dynamics of the Volatility Surface: Dynamics of the Volatility Skew under LV

It is obvious that the forward IV surface is substantially flatter than today's because the forward LV skews are all flatter. Contrast this with a SV model where IV skews are approximately time-homogeneous.

In other words, LV models imply that future B-S IV surfaces will be flat (relative to today's) and SV models imply that future B-S IV surfaces will look like today's.

Chapter 8: Dynamics of the Volatility Surface: Stochastic IV Models

Many authors including Brace, Goldys, Klebaner and Womersley (2001), Cont and da Fonseca (2002), Ledoit, Santa-Clara and Yan (2002) and Schonbucker (1999) have looked at models that allow the entire IV surface to diffuse. It turns out that if the undelying price process is assumed continuous (with no jumps), the statics and dynamics of the IV surface are highly constrained.

Chapter 8: Dynamics of the Volatility Surface: Stochastic IV Models

In particular, nondiscounted option prices are risk-neutral expectations of future cashflows and as such must be martingales. Changes in the call price reflect changes in the underlying and changes in IV. Imposing the martingale constraint $E[dC_t] = 0$ gives a tight relationship between the various sensitivities and many results such as equation (7.3) (B-S IV skew)

$$\frac{\partial \sigma_{BS}^2(x,t)}{\partial x} = \frac{\rho \eta}{2} \beta(v_0)$$

follow immediately from this.

Chapter 8: Dynamics of the Volatility Surface: Stochastic IV Models

More recently, Durrleman (2005) showed how to extract the dynamics of instantaneous variance from the dynamics of the observed IV surface in the limit of very short expirations and very close to at-the-money. Conversely, given a SV model, he showed how to deduce the shape of the IV surface in that same neighbourhood.

Chapter 8: Dynamics of the Volatility Surface: Stochastic IV Models

However, to get this impressive resuls, one has to assume continuity of the underlying price process but as we have seen earlier, jumps in the underlying are needed to explain the shape of the IV surface. Moreover, as noted by Cont, da Fonseca and Durrleman (2002) and as observed by any option trader, there appear to be jumps in the IV surface too.

Chapter 8: Dynamics of the Volatility Surface: Digital Options and Digital Cliquets (Digital Options)

Applying our insights to the valuation of actual derivative contracts, we choose to study digital options because their valuation involves the volatility skew directly.

Digital Options A digital (call) option $D(K,T)$ pays 1 if the stock price S_T at expiration T is greater than the strike price K and zero otherwise. It may be valued as the limit of a call spread as the spread between the strikes is reduced to zero:

$$D(K,T) = -\frac{\partial C(K,T)}{\partial K},$$

where $C(K,T)$ represents the price of a European call option with strike K expiring at time T.

Chapter 8: Dynamics of the Volatility Surface: Digital Options

To see that its is very sensitive to the volatility skew, we rewrite the European call price in equation above in terms of its B-S IV $\sigma_{BS}(K,T)$:

$$D(K,T) = -\frac{\partial C_{BS}(K,T,\sigma_{BS}(K,T))}{\partial K} = -\frac{\partial C_{BS}}{\partial K} - \frac{\partial C_{BS}}{\partial \sigma_{BS}}\frac{\partial \sigma_{BS}}{\partial K}.$$

To get an idea of the impact of the skew in practice, consider our usual idealized market with zero interest rate and dividents and a one-year digital option struck at-the-money.

Chapter 8: Dynamics of the Volatility Surface: Digital Options

Suppose further that at-the-money volatility is 25% and volatility skew (typical of SPX for example) is 3% per 10% change in strike. Its value is given by

$$\begin{aligned} D(1,1) &= -\frac{\partial C_{BS}}{\partial K} - \frac{\partial C_{BS}}{\partial \sigma_{BS}}\frac{\partial \sigma_{BS}}{\partial K} \\ &= N(-\frac{\sigma}{2}) - vega \times skew \\ &= N(-\frac{\sigma}{2}) + \frac{1}{\sqrt{2\pi}}e^{-(d_1^2)/2} \times 0.3 \\ &= N(-\frac{\sigma}{2}) + 0.4 \times 0.3. \end{aligned}$$

If we had ignored the skew contribution, we would have got the price of the digital option wrong by 12% of notional!

Chapter 8: Dynamics of the Volatility Surface: Digital Cliquets

Here is part of a definition of the word *cliquet* from the Dictionary of Financial Risk Management (Gastineau and Kritzman, 1999):

> *The French like the sound of 'cliquet' and seem prepared to apply the term to any remotely appropriate option structure. (1) Originally a periodic reset option with multiple payouts on a ratchet option (from vilbrequin a cliquet-ratchet brace). Also called Ratchet Option...*

Cliquet/Ratchet Options (Global Derivatives web)

A *cliquet or ratchet option* is a series of at the money options, with periodic settlement, resetting the strike value at the then current price level, at which time, the option locks in the difference between the old and new strike and pays that out as the profit. The profit can be accumulated until final maturity, or paid out at each reset date.

Cliquet/Ratchet Options (WikipediA)

A cliquet option or ratchet option is an exotic option consisting of a series of consecutive forward start options. The first is active immediately. The second becomes active when the first expires, etc. Each option is struck at-the-money when it becomes active.

Cliquet/Ratchet Options (WikipediA)

A cliquet is, therefore, a series of at-the-money options but where the total premium is determined in advance. A cliquet can be thought of as a series of "pre-purchased" at-the-money options. The payout on each option can either be paid at the final maturity, or at the end of each reset period.

Example of Cliquet Option

* A three-year cliquet with reset dates each year would have three payoffs. The first would payoff at the end of the first year and has the same payoff as a normal ATM option.

* The second year's payoff has the same payoff as a two-year option, but with the strike price equal to the stock price at the end of the first year.

* The third year's payoff has the same payoff as a three-year option, but with the strike price equal to the stock price at the end of the second year.

Chapter 8: Dynamics of the Volatility Surface: Digital Cliquets

Since the word is originally French, here is an elegant definition of the 'Effet-cliquet' from the French Web site http://lexique-financier.actufinance.fr

> *Mecanisme qui permety de figer une performance meme si l'actif correspondant baisse par la suite* (Mechanism that permits a profit to be locked in even if the underlying subsequently declines.)

The payoff of a hypothetical cliquet contract is shown in Figure 8.1.

Illustration of a Cliquet Payoff

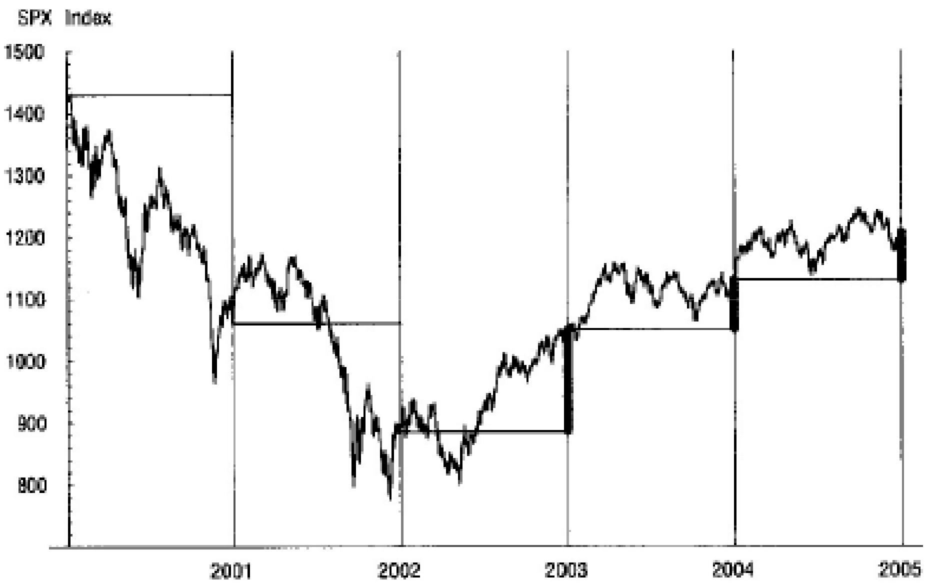


FIGURE 8.1 Illustration of a cliquet payoff. This hypothetical SPX cliquet resets at-the-money every year on October 31. The thick solid lines represent nonzero cliquet payoffs. The payoff of a 5-year European option struck at the October 31, 2000, SPX level of 1429.40 would have been zero.

Chapter 8: Dynamics of the Volatility Surface: Digital Cliquets

For our purpose, a cliquet is just a series of options whose strikes are set on a sequence of future dates. In particular, a digital cliquet is a sequence of digital options whose strikes will be set (usually) at the prevailing stock price on the relevant reset date.

Denoting the set of reset dates by $(t_1, t_2, ..., t_n)$, the digital cliquet pays

$$Coupon \times \theta(S_{t_i} - S_{t_{i-1}})$$

at t_i, where $\theta(\cdot)$ represents the Heaviside function.

Chapter 8: Dynamics of the Volatility Surface: Digital Cliquets

One can see immediately that the package consisting of a zero coupon bond together with a digital cliquet makes a very natural product for a risk-averse retail investor-he or she typically gets an above-market coupon if the underlying stock index is up for the period (usually a year) and a below-market coupon (usually zero) if the underlying stock index is down.

Not suprisingly, this product was and is very popular and as a result, many equity derivatives dealers have digital cliquet on their books.

Conclusion

1. Dynamics of the Volatility Skew under SV

2. Dynamics of the Volatility Skew under LV

3. Stochastic IV Models

4. Digital Options and Digital Cliquets

The End of Chapter 8

Thank You for Your Time and Attention!

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