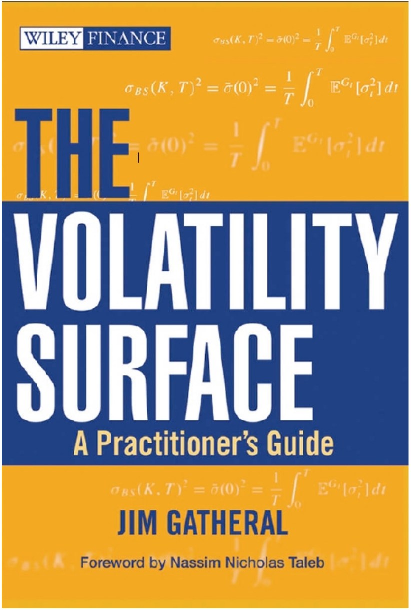

Chapter 9
Barrier Options

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Introduction

The goal of this chapter is to understand the relative pricing of various types of barrier options under stochastic and local volatility dynamics.

→ Important building blocks for structured products, but their valuation can be highly model dependent.

→ Developing intuition is particularly important not only to be able to estimate the value of a barrier options but also to know whether the output of a model should be trusted or not.

→ We suppose that European options of all strikes and expirations are traded in the market and our ovjective is to price barrier options consistently with these European option prices.

Definitions

Knock-out option

is an option that becomes worthless when a prespecified “barrier” level is reached.

Live-out option

is a special case of knock-out option that is significantly in-the-money when it knocks out.

Definitions

Knock-in option

is an option that can only be exercised if a barrier level is reached prior to exercise.

Rebate

is an amount of money paid to a barrier option buyer if the barrier is hit.

Limiting Cases

Limit Orders

The author assume that

- We sell a knock-out call option with barrier B equal to the strike price K below the current stock price S.
- We hedge this position by buying one stock per option and we charge $S_0 - K$ as the premium.
- If interest rates and dividends are zero, it is clear that this hedge is perfect.

Explaination

Two ways:

- Suppose first that the barrier is never hit: The buyer of the knock-out call option exercises the option and we deliver the stock. Net proceeds are

$$-(S_T - K) + (S_0 - K) + (S_T - S_0) = 0.$$

- If the barrier is hit, we lose $S_0 - K$ on our purchase of stock which is perfectly offset by the premium we charged.

Limiting Cases

Limit Orders

- as long as the stock price remains above the barrier level, we are net flat.
- when the barrier is hit, the option knocks out and we are left long of the stock we bought to hedge.
- This is exactly the portfolio we would have if the option buyer had left us a stop-loss order to see stock if the price ever reached the barrier level B.

Limiting Cases

Limit Orders

Difference between two contracts

- a barrier option guarantees execution at the barrier lever
- a conventional stop-loss order would get filled at the earliest opportunity after the barrier is hit

- there would be no difference between the two contracts if we could really trade continuously as models conventionally assume
- a knock-out option needs to be priced more highly than the model price to compensate for the risk of the stock price gapping through the barrier level.
- practitioners compensate for gap risk when pricing options by moving the barrier by some amount related to the expected gap in the stock price when the barrier is hit.

Limit Orders

Summary

- when $K = B < S_0$, the price of a knock-out call is given by the difference $S_0 - K$ between the current stock price and the strike price plus a bit to compensate for gap risk.
- if the strike price K and the barrier level B are not equal but not so far apart with $B \leq K < S_0$
  - neither gamma nor vega would be very high relative to the European option with the same strike K
  - nor would we expect the price of such a knock-out option to be very sensitive to the model used to value it

Limiting Cases

European Capped Calls

This option is a call struck at K with barrier $B > S_0$ such that if the stock price reaches B before expiration, the option expires and pays out intrinsic of $B - K$.

If the barrier is far away from the current stock price S_0, the price of such an option cannot be very different from the price of a conventional European option.

Consider a portfolio consisting of a long European option struck at K and short the capped call.

- if the barrier is not hit, this potfolio pays nothing.
- if the barrier is hit, the portfolio will be long a European option and short cash in the amount of the intrinsic value $B - K$.

Limiting Cases

The time value of this European option cannot be very high because

- by assumption, $B >> S_0$
- the barrier is most likely to be hit close to expiration

Since the value of the capped call must be close to the value of a conventional European call

- the value of the capped call cannot be very model dependent
- it should be well approximated by a model using Black-Scholes assumptions and the implied volatility of the cooresponding European option.

European Capped Calls

capped calls to live-out calls

we only need to omit the rebate at the barrier

- we then have a call option struck at K that goes deep-in-the-money as the stock price approaches the barrier $B >> K$ and knocks out when the stock price reaches B (with no rebate)
- so to get intuition for the pricing and hedging of live-out options, we need only study the pricing and hedging of the rebate.

The Reflection Principle

We suppose that the stock price is driven by a constant volatility stochastic process with zero log-drift. That is:

$$dx = \sigma dZ$$

with $x := \log(S/K)$.

In this special case, there is a very simple relationship between the price of a European binary option struck at B and the value of the one-touch option struck at B.

The Reflected Path

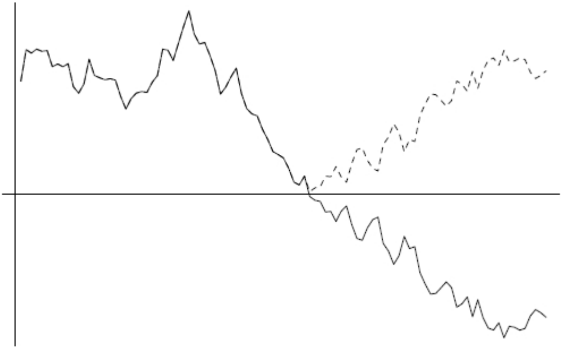

FIGURE 9.1 A realization of the zero log-drift stochastic process and the reflected path.

The Reflected Path

- the dashed path has the same probability of being realized as the original solid path.
- the probability of hitting the barrier B is exactly twice the probability of ending up below the barrier at expiration.
- the value of a one-touch option is precisely twice the value of a European binary put.

- an at-the-money barrier has 100% chance of getting hit but there is only 50% chance of ending up below the barrier at expiration in this special case.
- we might suppose that the ratio of the fair value of a one-touch option should be given by $B(S_0)^{-1}$ where $B(K)$ represents the value of a European binary put struck at K.

The Reflected Path

- for the Heston-Nandi model and parameters of Chapter 4 ($\nu = 0.04$, $\hat{\nu} = 0.04$, $\lambda = 10$, $\eta = 1$, $\rho = -1$)
- we may calculate $B(S_0) = 0.54614$
- the ratio of the one-touch price to the Eurppean binary price should be around $B(S_0)^{-1} = 1.831$.

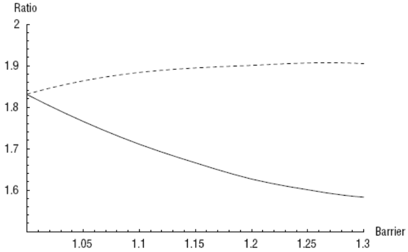

FIGURE 9.2 The ratio of the value of a one-touch call to the value of a European binary call under stochastic volatility and local volatility assumptions as a function of strike. The solid line is stochastic volatility and the dashed line is local volatility.

The Reflected Path

- As Taleb(1996) emphasizes, this ratio is very sensitive to modeling assumptions.
- Our guess is very accurate for the local volatility case.
- It is very inaccurate in the stochastic volatility case.
- Consider the effect of modeling assumptions on the price of a European binary call.

The Reflected Path

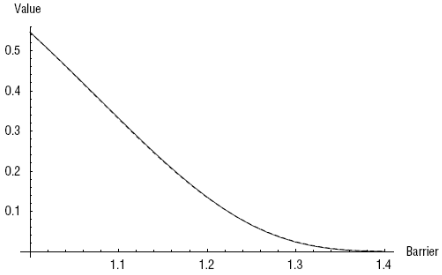

FIGURE 9.3 The value of a European binary call under stochastic volatility and local volatility assumptions as a function of strike. The solid line is stochastic volatility and the dashed line is local volatility. The two lines are almost indistinguishable.

- modeling assumptions have no effect
- the price of a European binary is independent of modeling assumptions
- depend only on the given prices of conventional European options

The Reflected Path

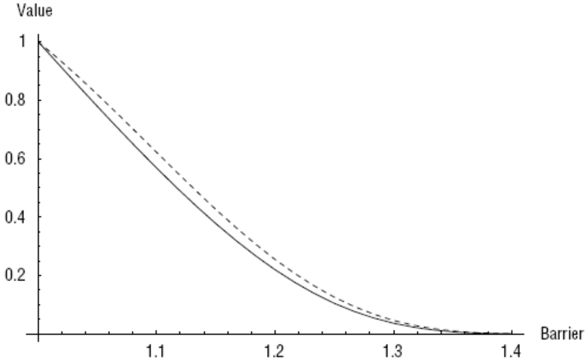

FIGURE 9.4 The value of a one-touch call under stochastic volatility and local volatility assumptions as a function of barrier level. The solid line is stochastic volatility and the dashed line is local volatility.

The Lookback Hedging Argument

Lookback Call Option

is an option that pays $(\tilde{S} - K)^+$ at expiration where $\tilde{S}$ is the maximum stock price over the life of the option and K is the strike price.

Assuming zero log-drift and constant volatility. Suppose we hedge a short position in this lookback call by holding two conventional European options struck at K.

- if the stock price never reaches K, both the lookback options and the European option expire worthless.
- if the stock price does reach K and increases by ΔK, the value of the lookback option must increase by ΔK.

The Lookback Hedging Argument

The new lookback option must pay $\Delta K + (\tilde{S} - (K + \Delta K))^+$—the payoff of another lookback option with a higher strike price plus a fixed cashflow ΔK.

Assuming we were right to hedge with two calls in the first place, the new hedge portfolio must be two calls struck at $K + \Delta K$. So we must rebalance our hedge portfolio by selling two calls struk at K and buying two calls struck at $K + \Delta K$. The profit generated by rebalancing is

$$\begin{aligned} 2C(K + \Delta K, K) - 2C(K + \Delta K, K + \Delta K) &\approx -2 \left.\frac{\partial C}{\partial K}\right|_{S=K} \Delta K \\ &= 2N(d_2)|_{S=K} \\ &= \Delta K \end{aligned}$$

The Lookback Hedging Argument

One-Touch Options again

Consider the value of a one-touch call option struck at B.

- it is the probability the maximum stock price is greater than B
- we can generate this payoff by taking the limit of a lookback call spread as the difference between the strikes gets very small
- because a lookback call has the same value as two European calls, a lookback call spread must have the same value as two European call spreads.
- a one-touch option is worth two European binary options when the log-drift is zero.

Put-Call Symmetry

We now assume zero interest rates and dividends and constant volatility again (as opposed to zero log-drift). In this case, by inspection of the Black-Scholes formula, we have

$$C\left(\frac{B^2}{S}, K\right) = \frac{K}{S} P\left(S, \frac{B^2}{K}\right)$$

Put-Call Symmetry

From one of the many references containing closed-form formulas for knock-out options, we may deduce that

$$\begin{aligned} DO(S,K,B) &= C(S,K) - \frac{S}{B} C\left(\frac{B^2}{S}, K\right) \\ &= C(S,K) - \frac{K}{B} P\left(S, \frac{B^2}{K}\right) \end{aligned}$$

where $DO(\cdot)$ represents that value of a down-and-out call.

Quasistatic Hedging and Qualitative Valuation

We cam generalize the static hedging procedure of the previous section to other cases where interest rates, dividents and volatility have arbitrary structure. Although there is no exact static hedge in the general case, we can construct a portfolio that has rather small payoffs under all reasonable scenarios.

A sophisticated version of this procedure known as the Lagranguan Uncertain Volatility Model is described by Avellaneda, Levy, and Parás in 1995. In this model

- volatility is bounded but uncertain
- volatility is assumed to be high when the portfolio is short gamma and low when the portfolio is long gamma.
- different prices are generated depending on whether an option position is long or short.

Quasistatic Hedging and Qualitative Valuation

We may use the quasi-static hedging idea to determine the impact of modeling assumptions on the valuation of any given barrier like claim.

- first determine what the quasistatic hedge looks look
- figure out how this hedge behaves under various future stock price and volatility scenarios

Quasistatic Hedging and Qualitative Valuation

Out-of-the-Money Barrier Options

Suppsoe we sell a call with strike K that knocks out at $B < K$. In this case, the quasistatic hedge portfolio is long a European call with strike K and short an out-of-the-money European put (with strike B^2/K in the zero log-drift case)

- if the barrier is never hit, the hedge is obviously good
- if the barrier is hit, the value of the then out-of-the-money European call and out-of-the-money European put in the hedge portfolio depend on the then volatility.

Quasistatic Hedging and Qualitative Valuation

Out-of-the-Money Barrier Options

- the skew will be flatter under local volatility than under stochastic volatility.
- we may conclude that when the barrier is hit, the hedge portfolio will be worth more under local volatility than under stochastic volatility.
- the skew effect is not very big because both options will be out-of-the-money
- We therefore guess that the pricing of a call that knocks out out-of-the-money would be a little lower under local volatility than under stochastic volatility.

Quasistatic Hedging and Qualitative Valuation

Out-of-the-Money Barrier Options

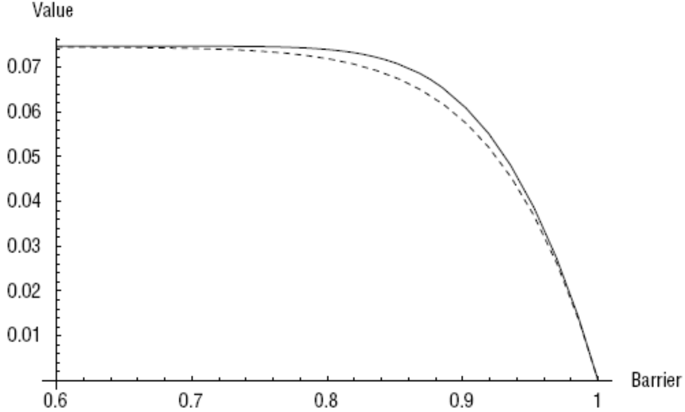

FIGURE 9.5 Values of knock-out call options struck at 1 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility.

Quasistatic Hedging and Qualitative Valuation

Out-of-the-Money Barrier Options

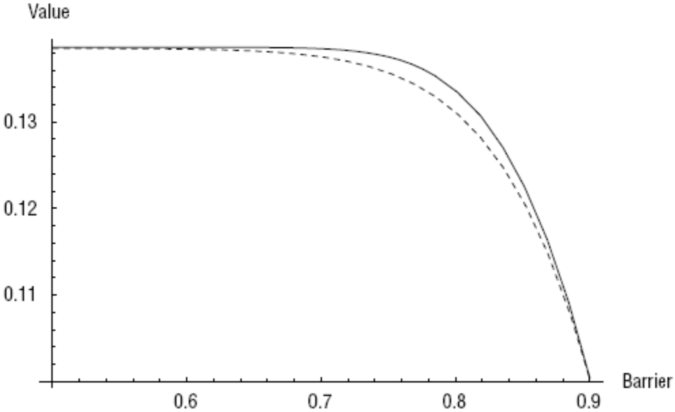

FIGURE 9.6 Values of knock-out call options struck at 0.9 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility.

Quasistatic Hedging and Qualitative Valuation

One-Touch Options

Recall out that a one-touch call option pays some prespecified amount of money if an upside barrier is hit.

Suppose we sell such a one-touch call option struck at $B > S$. The quasi-static hedge portfolio should be long a strip of European binary call options struck at B.

- if the barrier is never hit, the hedge is fine
- with the Heston-Nandi parameters, the skew when the barrier is hit will be flatter under local volatility than under stochastic volatility.

Quasistatic Hedging and Qualitative Valuation

One-Touch Options

- it follows that the binary call options will be worth more under stochastic volatility.
- on average the one-touch option costs less to hedge under stochastic volatility then under local volatility
- its upfront valuation should be lower under stochastic volatility.

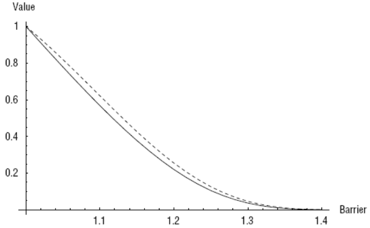

FIGURE 9.4 The value of a one-touch call under stochastic volatility and local volatility assumptions as a function of barrier level. The solid line is stochastic volatility and the dashed line is local volatility.

Quasistatic Hedging and Qualitative Valuation

Live-Out Options

We model a live-out call as a capped call minus a one-touch.

- we guess that the value of the capped call should be pretty much identical under stochastic and local volatility modeling assumption.
- we expect the value of the one-touch option to be lower under stochastic volatility than under local volatility.
- we conclude that with these parameters, a live-out call should be worth more under stochastic volatility than under local volatility.
- note that the difference in valuation between the two modeling assumptions can be very substantial.

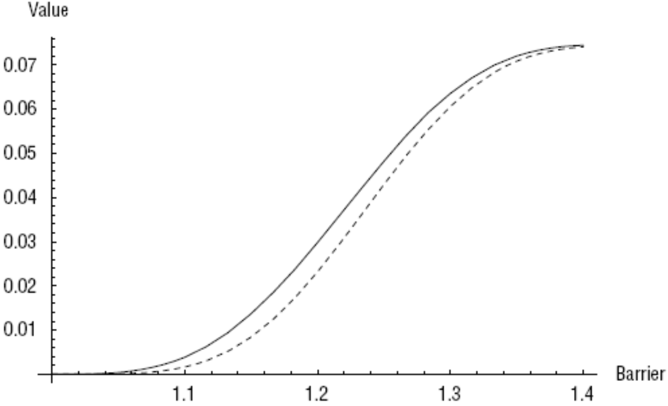

FIGURE 9.7 Values of live-out call options struck at 1 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility.

Quasistatic Hedging and Qualitative Valuation

Lookback Options

Suppose now we sell a lookback call option struck at $K > S$.

The hedge is roughly two European call options struck at K.

- if the stock price never reaches K, the hedge is fine
- if and when the stock price reaches K, we need to sell our two European calls with strike K and buy two new calls with strike $K + \Delta K$.
- we conclude that with these parameters, a lookback call should be worth less under stochastic volatility than under local volatility.

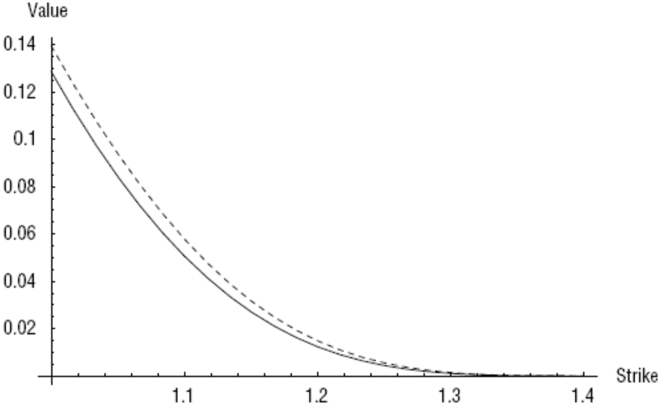

FIGURE 9.8 Values of lookback call options as a function of strike. The solid line is stochastic volatility; the dashed line is local volatility.

Adjusting for Discrete Monitoring

Theorem (Broadie, Glasserman, and Kow in 1999)

Let $V(B)$ be the price of a continuous barrier option, and $V_m(B)$ be the price of an otherwise identical barrier option with m monitoring points. Then we have the approximation

$$V_m(B) = V(Be^{\pm\beta\sigma\sqrt{T/m}}) + o(1/\sqrt{m})$$

with $+$ for an up option and $-$ for a down option, where the constant

$$\beta = -\frac{\zeta(1/2)}{\sqrt{2\pi}} \approx 0.5826.$$

and ζ is the Riemann zeta function.

Adjusting for Discrete Monitoring

Discretely Monitored Lookback Options

The expected difference between the continuous maximum and the discrete maximum might be approximated by the same number. Broadie, Glasserman and Kou in 1999 showed that

$$\mathbb{E}[\hat{S}_T] = \mathbb{E}[\tilde{S}_T]e^{-\beta\sigma\sqrt{\Delta T}}$$

where $\hat{S}$ is the discrete maximum, $\tilde{S}$ is the continuous maximum and ΔT is the monitoring interval.

Adjusting for Discrete Monitoring

Discretely Monitored Lookback Options

An at-the-money lookback option $\hat{L}(S_0, T)$ pays the discrete maximum minus the initial stock price. We then have

$$\begin{aligned}\hat{L}(S_0, T) &= \mathbb{E}[\hat{S}_T - S_0] \\ &= \mathbb{E}[\tilde{S}_T] e^{-\beta\sigma\sqrt{\Delta T}} - S_0 \\ &= \tilde{L}(S_0 e^{-\beta\sigma\sqrt{\Delta T}}, T) - S_0(1 - e^{-\beta\sigma\sqrt{\Delta T}})\end{aligned}$$

where $\tilde{L}$ is the equivalent continuously monitored option.

Parisian Options

Barrier Wars

Continuously monitored barrier options can tempt either the option buyer or seller to influence the underlying stock price.

One way to retain the benefits of barrier options while minimizing the risk of manipulation is to specify that in order for the option to knock-in(or out), the underlying stock price must stay outside the barrier for a minimum period of time referred to as a *window*.

In this case, the option is called a *Parisian* option.

Parisian Options

- Parisian-style features are common in convertible bonds and whenever the size of a derivative is large relative to the liquidity of the underlying stock.
- With the usual constant parameter assumptions, Parisian options can be valued in almost-closed form by applying some results in the excursion process of Brownian motion.
- As in most other barrier option valuation problems, the most natural valuation approach is numerical PDE as described for example by Tavella and Randall(2000).

Some Applications of Barrier Options

Ladders

Consider s stripe of capped calls with strikes B_i strictly increasing and greater than the initial stock price S_0.

- The cap of the option with strike B_i is B_{i+1}
- When the barrier is hit at B_{i+1}, a rebate $B_{i+1} - B_i$ is paid
- The buyer of such an option locks in his gain each time a barrier is crossed.
- In the limit where the caps are very close to the strikes, a ladder approximates a lookback option
- the value of the ladder would be approximately twice the value of a European option.

Some Applications of Barrier Options

Ranges

Ranges

- is another popular investment
- is one that pays a high coupon for each day that the stock price remains within a range
- The buyer of such an option locks in his gain each time a barrier is crossed.
- but ceases paying a coupon as soon as one of the boundaries is hit.
- This is just a one-touch double barrier constriction.

Conclusion

- Barrier option values can be very sensitive to modeling assumptions and prices must be adjusted to take this into account.
- By understanding limiting cases, we can gain a good qualitative understanding of the appropriate valuation and hedge portfolio for any given barrier option.
- Market practitioners are often reluctant to quote on any barrier option given the potential valuation uncertainty and hedging complexity.
- Sometimes a barrier option is much less risky and easier to price than its European equivalent.