

Pricing and Hedging of Commodity Derivatives using the Fast Fourier Transform

Vladimir Surkov

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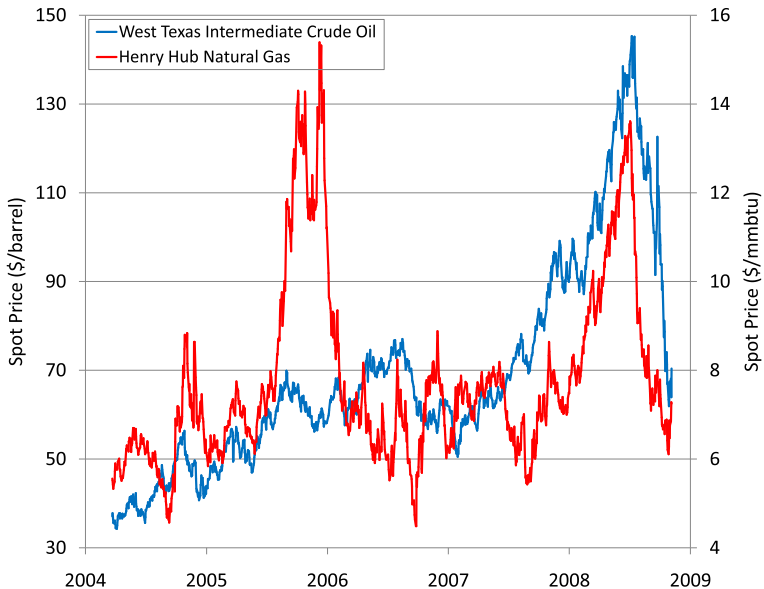
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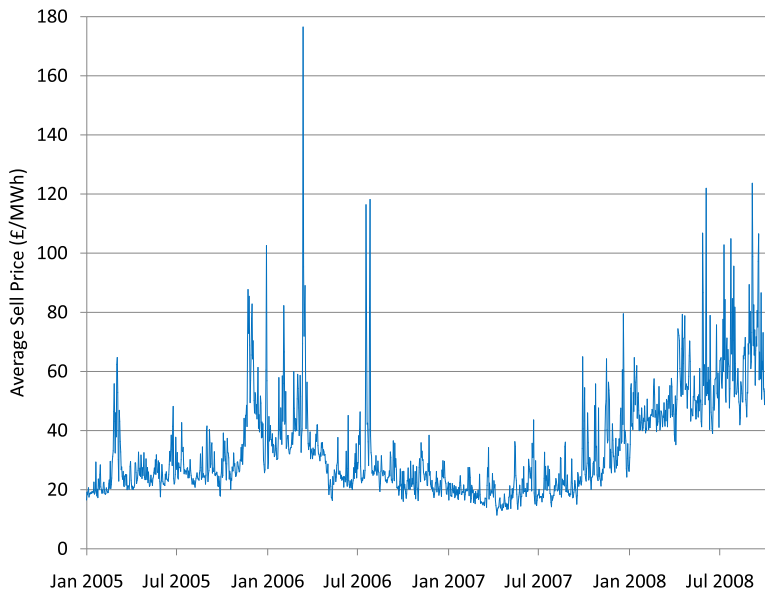
- 1 Generalized Model for Commodity Spot Prices
- 2 Fourier Space Time-stepping framework
- 3 Computing Option Greeks
- 4 Dynamic and Static Hedging

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WTI Crude Oil and Henry Hub Gas



Great Britain System Sell Prices



Generalized Model for Commodity Spot Prices

Commodity prices

- Exhibit high volatilities and spikes and prices
- Tend to revert to long run equilibrium prices

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The Model

$$d\mathbf{X}(t) = (\Theta(t) - \kappa\mathbf{X}(t)) dt + d\mathbf{W}(t) + d\mathbf{J}(t)$$
$$\mathbf{S}(t) = \mathbf{S}(0) \exp\{\mathbf{X}(t)\}$$

Generalized Model for Commodity Spot Prices

Commodity prices

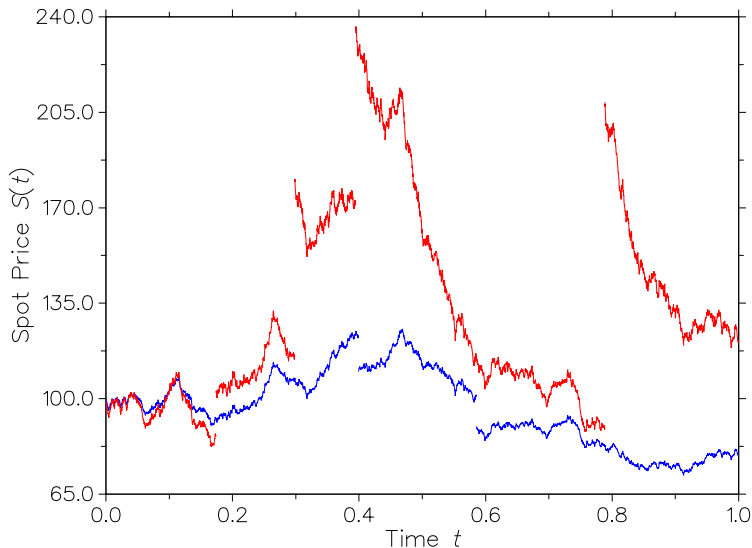
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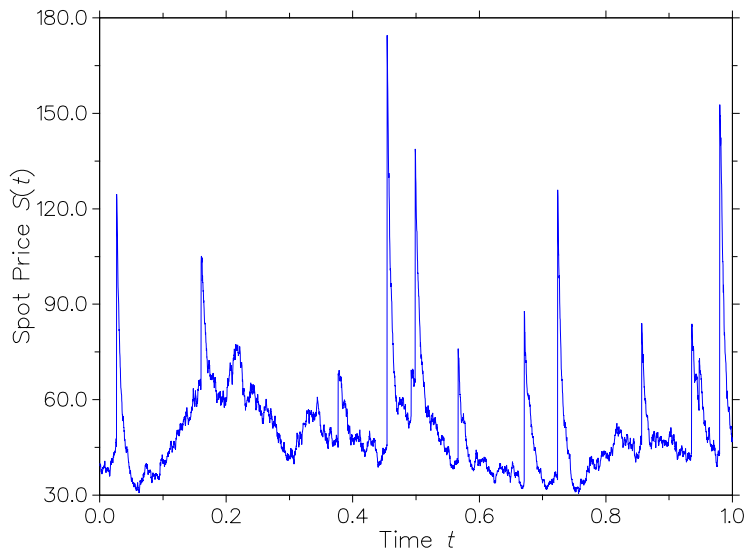
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- Multi-factor, mean-reverting model with jumps
- Generalizes a number of well-known models, such as Gibson and Schwartz (1990), Clewlow and Strickland (2000), Hikspoors and Jaimungal (2007)
- Mean-reversion rate is time dependent - allows to incorporate seasonality

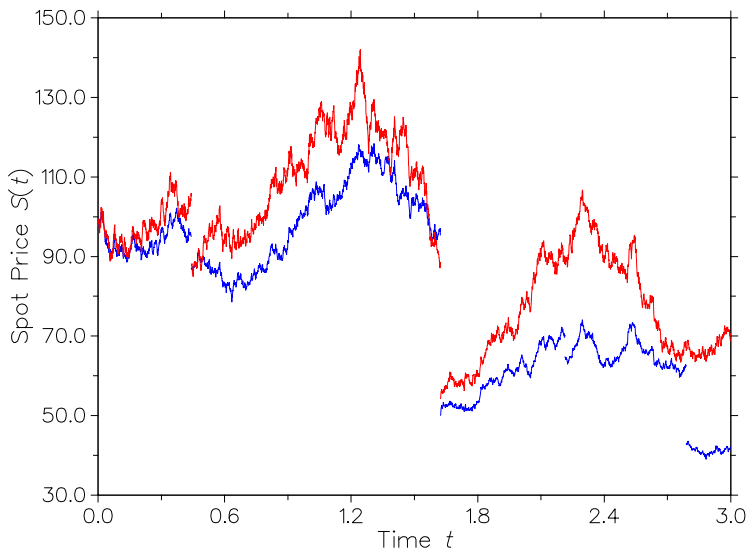
One-Factor Mean-Reverting Process with Jumps



Two-Factor MR Process with Different Decay Rates



One-Factor MR Processes with Co-dependent Jumps



Numerical Methods for Option Pricing

- Monte Carlo methods
- Tree methods
- Finite difference methods
 - Alternating Direction Implicit-FFT - Andersen and Andreasen (2000)
 - Implicit-Explicit (IMEX) - Cont and Tankov (2004)
 - IMEX Runge-Kutta - Briani, Natalini, and Russo (2004)
 - Fixed Point Iteration - d'Halluin, Forsyth, and Vetzal (2005)
- Quadrature methods
 - Reiner (2001)
 - QUAD - Andricopoulos, Widdicks, Duck, and Newton (2003)
 - Q-FFT - O'Sullivan (2005)
- Transform-based methods
 - Carr and Madan (1999)
 - Raible (2000)
 - Lewis (2001)
 - Lord, Fang, Bervoets, and Oosterlee (2008)

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Fourier Space Time-stepping Framework Overview

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
- Solve the resulting ODE analytically
- Utilize FFT to efficiently switch between real and Fourier spaces

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A framework for numerical pricing of financial derivatives

- Fast and precise pricing of a wide range of European and path-dependent, single- and multi-asset, vanilla and exotic derivatives
- Efficient handling of path-independent and discretely monitored derivatives
- Generic handling of different spot-price models and option payoffs

Pricing Framework in Real Space

- Option price at time t is the discounted expected future payoff

$$V(t, \mathbf{S}(t)) = e^{-r(T-t)} \mathbb{E}[\varphi(\mathbf{S}(T))]$$

Pricing Framework in Real Space

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$$V(t, \mathbf{S}(t)) = e^{-r(T-t)} \mathbb{E}[\varphi(\mathbf{S}(T))]$$

- The discount-adjusted and log-transformed price process $v(t, \mathbf{X}(t)) \triangleq e^{r(T-t)} V(t, \mathbf{S}(t)) e^{\mathbf{X}(t)}$ satisfies a PIDE

$$\begin{cases} (\partial_t + \mathcal{L}(t, \mathbf{x}) - \kappa \mathbf{x}' \partial_{\mathbf{x}}) v(t, \mathbf{x}) & = 0, \\ v(T, \mathbf{x}) & = \varphi(\mathbf{S}(0) e^{\mathbf{x}}) \end{cases}$$

where \mathcal{L} acts on twice-differentiable functions $g(\mathbf{x})$ as follows:

$$\mathcal{L}(t, \mathbf{x})g(\mathbf{x}) = (\boldsymbol{\Theta}(t)' \partial_{\mathbf{x}} + \frac{1}{2} \partial_{\mathbf{x}}' \boldsymbol{\Sigma} \partial_{\mathbf{x}}) g(\mathbf{x}) + \int_{\mathbb{R}^n} (g(\mathbf{x} + \mathbf{y}) - g(\mathbf{x})) \nu(d\mathbf{y})$$

Fourier Transform

- A function in the space domain $g(\mathbf{x})$ can be transformed to a function in the frequency domain $\hat{g}(\boldsymbol{\omega})$, where $\boldsymbol{\omega}$ is given in radians per second, and vice-versa using the continuous Fourier transform

$$\mathcal{F}[g](\boldsymbol{\omega}) \triangleq \int_{-\infty}^{\infty} g(\mathbf{x}) e^{-i\boldsymbol{\omega}'\mathbf{x}} d\mathbf{x}$$
$$\mathcal{F}^{-1}[\hat{g}](\mathbf{x}) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\boldsymbol{\omega}) e^{i\boldsymbol{\omega}'\mathbf{x}} d\boldsymbol{\omega}$$

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- Continuous Fourier transform is a linear operator that maps spatial derivatives ∂_x into multiplications in the frequency domain

$$\mathcal{F}[\partial_x^n g](\boldsymbol{\omega}) = i\boldsymbol{\omega} \mathcal{F}[\partial_x^{n-1} g](\boldsymbol{\omega}) = \dots = (i\boldsymbol{\omega})^n \mathcal{F}[g](\boldsymbol{\omega})$$

Pricing Framework in Fourier Space

- Applying the Fourier transform to the pricing PDE we obtain a PDE in frequency space

$$\begin{cases} \left(\partial_t + \hat{\mathcal{L}}(t, \omega) + \kappa + \kappa\omega\partial_\omega \right) \hat{v}(t, \omega) = 0, \\ \hat{v}(T, \omega) = \hat{\Phi}(T, \omega) \end{cases}$$

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- The Fourier transform of the operator $\mathcal{L}(t, \mathbf{x})$ can be computed analytically

$$\hat{\mathcal{L}}(t, \omega) = i\omega\Theta(t) - \frac{1}{2}\omega'\Sigma\omega + \int \left(e^{i\omega'z} - 1 \right) \nu(dz)$$

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- Introduce a new coordinate system via frequency scaling

$$\tilde{v}(t, \omega) = \hat{v}(t, e^{\kappa'(t-t_*)} \omega)$$

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- Introduce a new coordinate system via frequency scaling

$$\tilde{v}(t, \omega) = \hat{v}(t, e^{\kappa'(t-t_*)} \omega)$$

- The PDE reduces to an ODE in time parameterized by ω

$$\begin{cases} \left(\partial_t + \tilde{\mathcal{L}}(t, \omega) + \kappa \right) \tilde{v}(t, \omega) & = 0, \\ \tilde{v}(T, \omega) & = \tilde{\Phi}(T, \omega) \end{cases}$$

Pricing Framework in Fourier Space (cont.)

- Given the value of $\tilde{v}(t, \omega)$ at time $t_2 \leq T$, the constant coefficient ODE is easily solved to find the value at time $t_1 < t_2$:

$$\tilde{v}(t_1, \omega) = \tilde{v}(t_2, \omega) \cdot e^{\tilde{\Psi}_{\kappa}(t_1, \omega; t_2)},$$

where the frequency space propagator is

$$\tilde{\Psi}_{\kappa}(t_1, \omega; t_2) = \int_{t_1}^{t_2} \tilde{\mathcal{L}}(s, \omega) ds + \text{Tr } \kappa(t_2 - t_1)$$

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- The solution in terms of original coordinates (with $t_{\star} = t_1$) is given by

$$\hat{v}(t_1, \omega) = \hat{v}(t_2, e^{\kappa'(t_2-t_1)} \omega) \cdot e^{\hat{\Psi}_{\kappa}(t_1, \omega; t_2)}$$

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Pricing Framework in Fourier Space (cont.)

- The scaled option prices in frequency space can be obtained from the scaled option prices in real space

$$\mathcal{F}[g](t, e^{\kappa'(t_2-t_1)} \omega) = \mathcal{F}[\check{g}](t, \omega) \cdot e^{-\text{Tr} \kappa (t_2-t_1)},$$

where $\check{g}(t, \mathbf{x}) \triangleq g(t, \mathbf{x} e^{-\kappa'(t_2-t_1)})$

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where $\check{g}(t, \mathbf{x}) \triangleq g(t, \mathbf{x} e^{-\kappa'(t_2-t_1)})$

- The final solution becomes

$$v(t_1, \mathbf{x}) = \mathcal{F}^{-1} \left[\mathcal{F}[\check{v}](t_2, \boldsymbol{\omega}) \cdot e^{\hat{\Psi}(t_1, \boldsymbol{\omega}; t_2)} \right] (\mathbf{x})$$

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mrFST Method for Propagating Option Prices

$$\mathbf{v}_{m-1} = \text{FFT}^{-1} \left[\text{FFT}[\check{\mathbf{v}}_m] \cdot e^{\hat{\Psi}(t_{m-1}, \omega; t_m)} \right]$$

Fourier Space Time-stepping Numerical Method

- European options

$$\mathbf{v}_0 = \text{FFT}^{-1} \left[\text{FFT} [\check{\mathbf{v}}_1] \cdot e^{\hat{\Psi}(t, \omega; T)} \right]$$

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- Bermudan/American options

$$\mathbf{v}_{m-1}^* = \text{FFT}^{-1} \left[\text{FFT} [\check{\mathbf{v}}_m] \cdot e^{\hat{\Psi}(t_{m-1}, \omega; t_m)} \right],$$

$$\mathbf{v}_{m-1} = \max \{ \mathbf{v}_{m-1}^*, \mathbf{v}_M \},$$

where \mathbf{v}_{m-1}^* represents the holding value of the option

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- Barrier options

$$\mathbf{v}_{m-1} = \text{FFT}^{-1} \left[\text{FFT} [\check{\mathbf{v}}_m] \cdot e^{\hat{\Psi}(t_{m-1}, \omega; t_m)} \right] \cdot \mathbb{1}_{\{x < B\}} + R \cdot \mathbb{1}_{\{x \geq B\}}$$

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- Exotic options, such as swings, can also be handled

Discrete Barrier Option Results

N	M	Value	Change	Convergence	Time (sec.)
2048	252	2.75818698			0.048
4096	252	2.77495289	0.0167659		0.099
8192	252	2.77164607	0.0033068	2.3420	0.210
16384	252	2.77315499	0.0015089	1.1319	0.523
32768	252	2.77395701	0.0008020	0.9118	0.974

- *Option*: Down-and-out barrier put $S = 100, K = 105, T = 1, B = 90, R = 3$ with daily monitoring
- *Model*: Merton jump-diffusion with mean reversion
 $\sigma = 0.2, \lambda = 1.0, \tilde{\mu} = -0.1, \tilde{\sigma} = 0.25, \theta = 90.0, \kappa = 0.75, r = 0.05$
- *Monte Carlo*: 2.77533300 – 95% CI width of 0.00323116 @ 114 sec.

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Computation of Greeks - State Variables

$$\text{Delta} - \partial_{S_k} v(t, \mathbf{x}) = \partial_{\mathbf{x}_k} v(t, \mathbf{x}) / \mathbf{S}_k(t)$$

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- Differentiation in real space computed via scaling in Fourier space

$$\partial_{\mathbf{x}_k} v(t, \mathbf{x}) = \mathcal{F}^{-1} [i\omega_k \cdot \hat{v}(t, \omega)](\mathbf{x}).$$

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- The discrete method for computing Deltas is then given by

$$\Delta_{k,m-1} = \text{FFT}^{-1} [i\omega_k \cdot \hat{\mathbf{v}}_{m-1}]$$

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- Obtained directly from the pricing ODE

$$\partial_t \tilde{v}(t, \boldsymbol{\omega}) = -(\tilde{\mathcal{L}}(t, \boldsymbol{\omega}) + \kappa) \tilde{v}(t, \boldsymbol{\omega})$$

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- The discrete method for computing Theta is then given by

$$\Theta_{m-1} = \text{FFT}^{-1} [-(\hat{\mathcal{L}}(t, \boldsymbol{\omega}) + \boldsymbol{\kappa}) \cdot \hat{\mathbf{v}}_{m-1}]$$

Computation of Greeks - Model Parameters

- In Fourier space, the sensitivity satisfies an ODE with source term

$$\partial_{\star} \left\{ (\partial_t + \tilde{\mathcal{L}}_{\kappa}) \tilde{v}(t, \omega) \right\} = (\partial_t + \tilde{\mathcal{L}}_{\kappa}) \partial_{\star} \tilde{v}(t, \omega) + \partial_{\star} \tilde{\mathcal{L}}_{\kappa} \cdot \tilde{v}(t, \omega) = \mathbf{0}$$

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- The ODE can be solved explicitly

$$\partial_{\star} v(t, \mathbf{x}) = \mathcal{F}^{-1} \left[\partial_{\star} \hat{\Psi}_{\kappa}(t, e^{\kappa'(T-t)} \omega; T) \cdot \hat{v}(t, \omega) \right] (\mathbf{x})$$

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- The discrete method for computing the sensitivity is then given by

$$\nabla_{\star, m-1} = \text{FFT}^{-1} \left[\partial_{\star} \hat{\Psi}_{\kappa}(t_{m-1}, e^{\kappa' \Delta t_m} \omega; t_m) \cdot \hat{v}_{m-1} \right]$$

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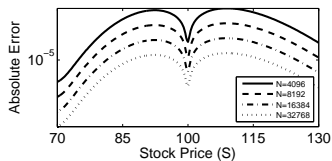
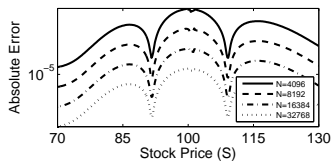
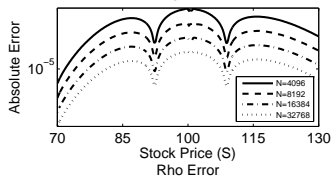
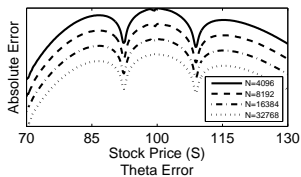
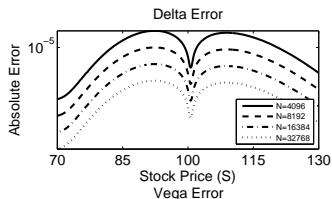
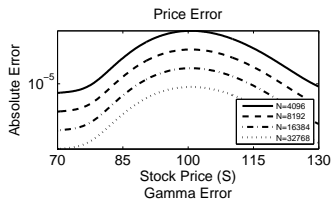
$$\partial_{\star} v(t, \mathbf{x}) = \mathcal{F}^{-1} \left[\partial_{\star} \hat{\Psi}_{\kappa}(t, e^{\kappa'(T-t)} \omega; T) \cdot \hat{v}(t, \omega) \right] (\mathbf{x})$$

- The discrete method for computing the sensitivity is then given by

$$\nabla_{\star, m-1} = \text{FFT}^{-1} \left[\partial_{\star} \hat{\Psi}_{\kappa}(t_{m-1}, e^{\kappa' \Delta t_m} \omega; t_m) \cdot \hat{\mathbf{v}}_{m-1} \right]$$

- Higher order derivatives computed in similar manner

Greeks Computation Errors



- 1 Generalized Model for Commodity Spot Prices
- 2 Fourier Space Time-stepping framework
- 3 Computing Option Greeks
- 4 Dynamic and Static Hedging

Dynamic Hedging

- Hedging portfolio for the option V consists of B units of cash, e units of the underlying asset S and N hedging instruments \vec{I} with weights $\vec{\phi}$

$$\Pi = \vec{\phi} \cdot \vec{I}(t, \mathbf{S}(t)) + e\mathbf{S}(t) + B - V(t, \mathbf{S}(t))$$

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- What about large movements?

Static Hedging - Minimization of Portfolio Variance

- Minimize portfolio price variance under expected asset price movement

$$\arg \min_{e_n, \vec{\phi}_n} \xi \mathbb{E}_{t_n} \left[\vec{\phi}_n \cdot \Delta \vec{I}_n + e_n \Delta S_n - \Delta V_n \right]^2 + (1 - \xi) \Upsilon_n.$$

where Υ_n is the transaction cost to rebalance the portfolio:

$$\Upsilon_n = \sum_{k=1}^N \left[\vec{\alpha}_k (\vec{\phi}_{k,n} - \vec{\phi}_{k,n-1}) \vec{I}_{k,n} \right]^2 + \left[\beta (e_n - e_{n-1}) S_n \right]^2,$$

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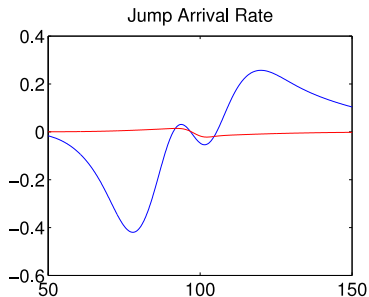
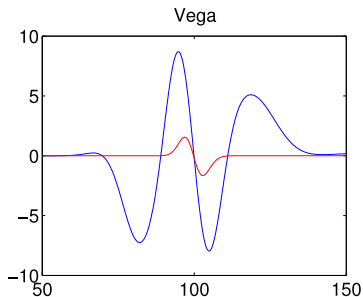
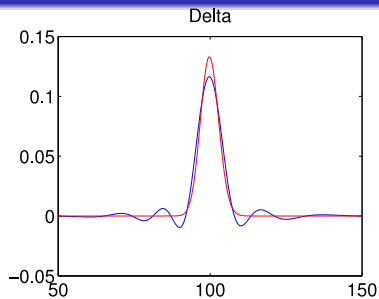
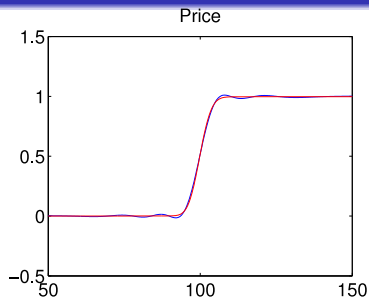
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- Since the objective function is quadratic, the optimality requires

$$\frac{\partial F}{\partial \phi_{k,n}} = \xi \mathbb{E}_{t_n} \left[(\vec{\phi} \cdot \Delta \vec{I} + e \Delta S - \Delta V) (2 \Delta I_k) \right] + (1 - \xi) \partial_{\phi_{k,n}} \Upsilon_n = 0$$

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Static Hedging - Minimization of Portfolio Variance



Static Hedging - Minimize Price and Greeks Variance

- Minimize portfolio price and Greeks variance under expected asset price movement

$$\arg \min_{e_n, \vec{\phi}_n} \xi \sum_{\mathcal{D}} w_{\mathcal{D}} \mathbb{E}_{t_n} \left[\vec{\phi}_n \cdot \Delta(\mathcal{D}\vec{I}_n) + e_n \Delta(\mathcal{D}S_n) - \Delta(\mathcal{D}V_n) \right]^2 + (1 - \xi) \Upsilon_n$$

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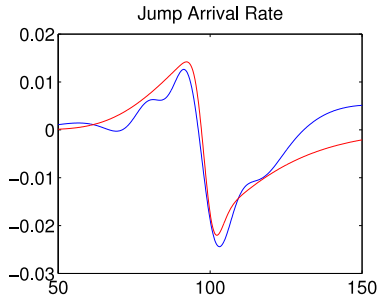
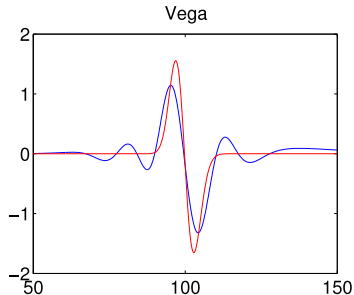
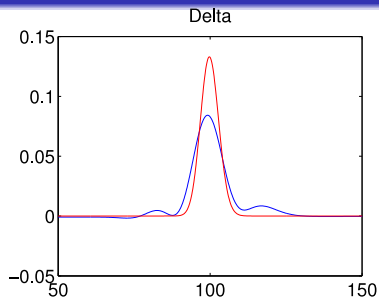
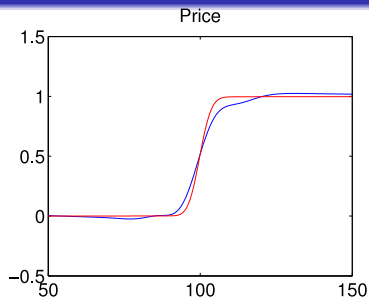
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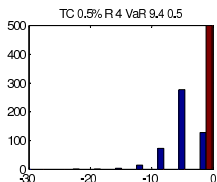
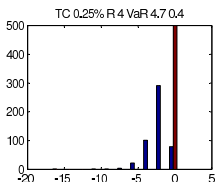
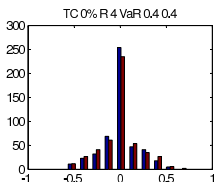
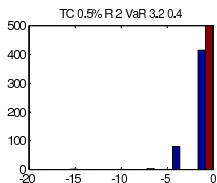
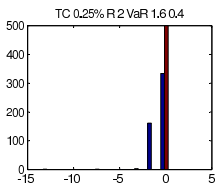
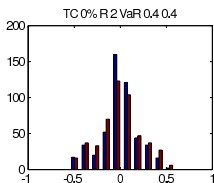
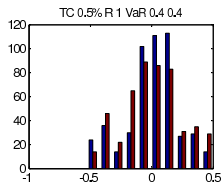
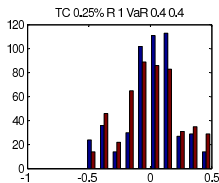
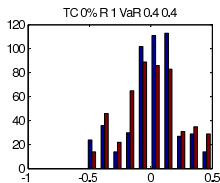
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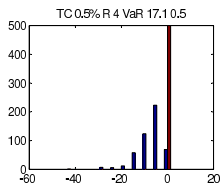
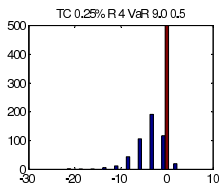
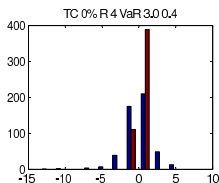
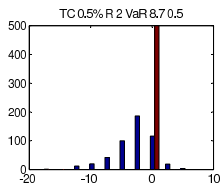
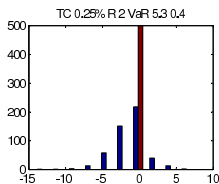
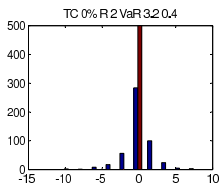
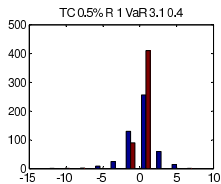
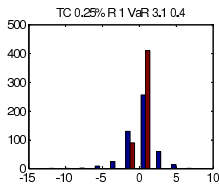
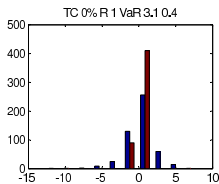
Static Hedging - Minimize Price and Greeks Variance



Loss Distribution and VaR - Constant Volatility



Loss Distribution and VaR - Dynamic Volatility



FST Framework Summary

The Approach

- Consider the PIDE for the option price
- Transform the PIDE into ODE in Fourier space
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 - Second order convergence in space and second order convergence in time for American options with penalty method

Thank You!

-  Jackson, K. R., S. Jaimungal, and V. Surkov (2008).
Fourier space time-stepping for option pricing with Lévy models.
Journal of Computational Finance 12(2), 1–28.
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-  Surkov, V. (2009).
Efficient static hedging under generalized Lévy models.
Working paper.

More at <http://ssrn.com/author=879101>