Information reduction by level-crossings in a credit risk model

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How do we model information in probability theory?

By σ-algebras.

How do we capture the dynamics of information?

By filtrations.

$\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is a collection of σ-algebras on a probability space $(\Omega, \mathcal{G}, P)$ such that:

$$\mathcal{G}_s \subset \mathcal{G}_t \subset \mathcal{G} \text{ for } s \leq t.$$

Credit Risk:

A risky bond promises paying the owner 1 dollar at time T (maturity). The promise may not be fulfilled if the company goes bankrupt before T.

Question: How do we quantify the risk of bankruptcy at a given time $t \in [0, T]$?

Let τ be the time of default.

The payoff at $T = 1_{\tau > T}$.

A dynamic probabilistic model for τ is required.

$\mathbb{F}$: Available information relevant to the default time.

conditional hazard rate:

$$\lambda(t) = \lim_{h \to 0} \frac{P(\tau \in (t, t+h] | \mathcal{F}_t)}{h}$$

The simplest model (the most reduced form model):

Let $N_t := 1_{\{\tau \le t\}}$ (one-point-process).

$\mathcal{F}_t = \sigma(N_s, s \le t)$.

i.e., we only know whether the company has gone bankrupt. Then conditional hazard rate is simply the hazard rate of τ, i.e.

$$\frac{f(t)}{1 - F(t)} 1_{\{\tau > t\}}$$

More structure:

Model the company's value with a continuous Markov process X.

τ: Hitting time of X to a default barrier.

$\mathcal{F}_t = \sigma(X_s, s \leq t)$ (natural filtration of X)

Drawback: Conditional hazard rate of τ is zero. Not consistent with market data.

Solution: Information reduction models

• (Duffie and Lando) Market observes X corrupted with noise.

• (Jarrow and Protter) Model the information available to the market with a *subfiltration* $\mathbb{F}$ of $\mathbb{G}$.

Definition: $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a subfiltration of $\mathbb{G}$ if $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$.

Model:

X: A non-singular diffusion with state space $I \subset \mathbb{R}$ with infinitesimal generator

$$\mathcal{A} \;:=\; \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}$$

$-\infty = x_0 < x_1, \ldots, x_N < x_{N+1} = \infty$: a finite collection of points in $\mathbb{R}$.

x_1: Default barrier

Define

$$R(x) = i \text{ if } x_i \leq x < x_{i+1}$$

$\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the right continuous and complete filtration generated by $R(X) = (R(X_s))_{s \geq 0}$.

What does $\mathbb{F}$ contain?:

• The first hitting times, τ_i of X of the x_i.

• Last exit times from $\mathcal{L}$. In particular, $g_t = \sup\{s \leq t, X_s \in \mathcal{L}\}$ and $U_t = t - g_t$ are both adapted to $\mathbb{F}$.

More generally,

• excursion intervals of X away from $\mathcal{L}$.

Since X has continuous paths and $\mathcal{L}$ is a closed set, $M = \{t \geq 0 : X_t \in \mathcal{L}\}$ is closed. Therefore

$$M^c = \cup_{n=1}^{\infty}(a_n, b_n)$$

(a_n, b_n) are called excursion intervals away from $\mathcal{L}$.

Main Result:

We derived two key quantities:

1. Conditional hazard rates of τ_i

2. Conditional default probability

Conditional default probability:

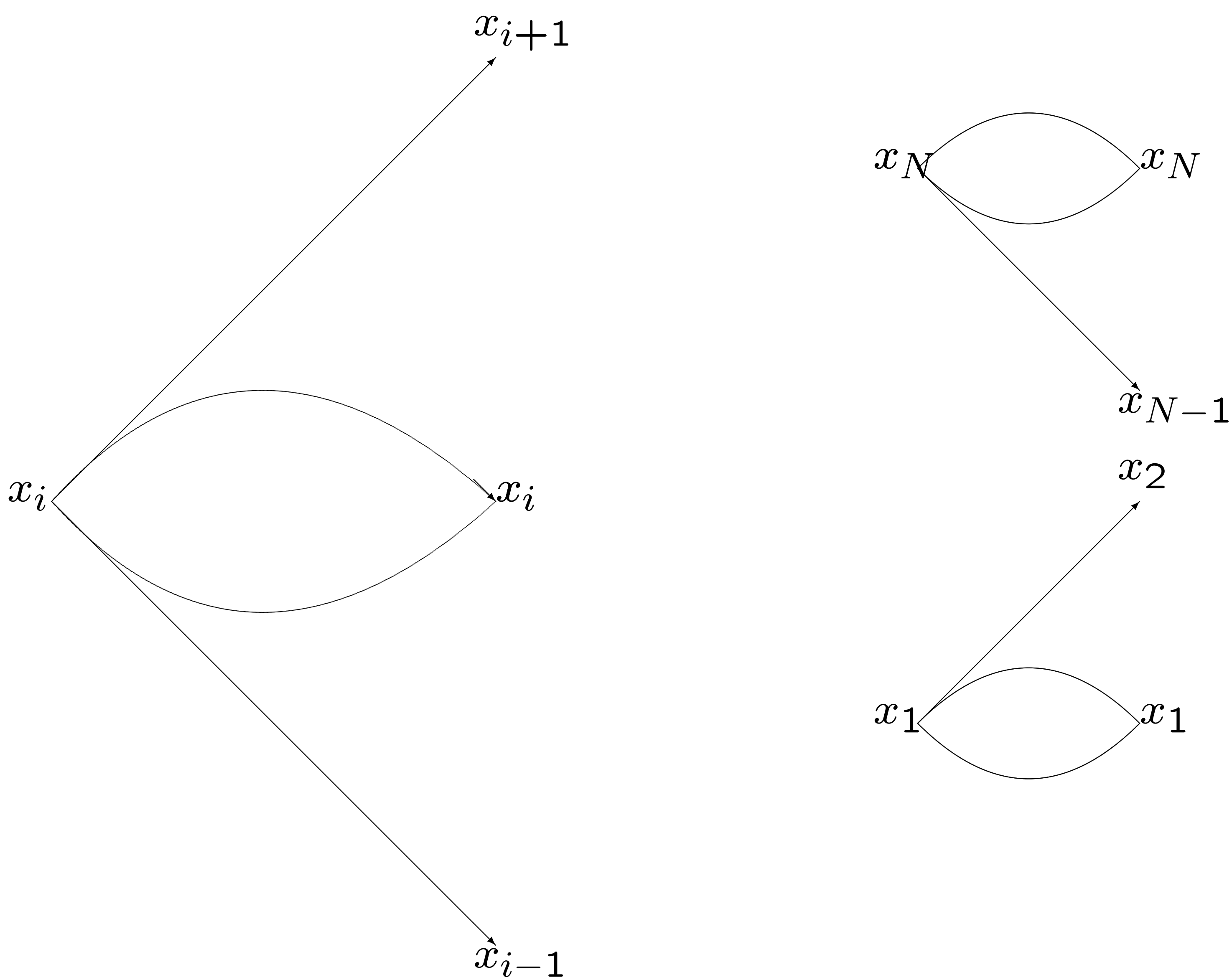
$$Y_t = E[1_{\{\tau \le T\}} | \mathcal{F}_t]$$

If $v(t,T)$, is the price of a simple bond at time t, then

$$v(t,T) = [1 - Y_t] e^{-\int_t^T r_s ds}$$

The main ingredient of the formulae:

Lévy measures F_i^{1+}, F_i^{1-} and F_i^{0+}, F_i^{0-}.


x_{i+1}
x_N
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x_1
x_1
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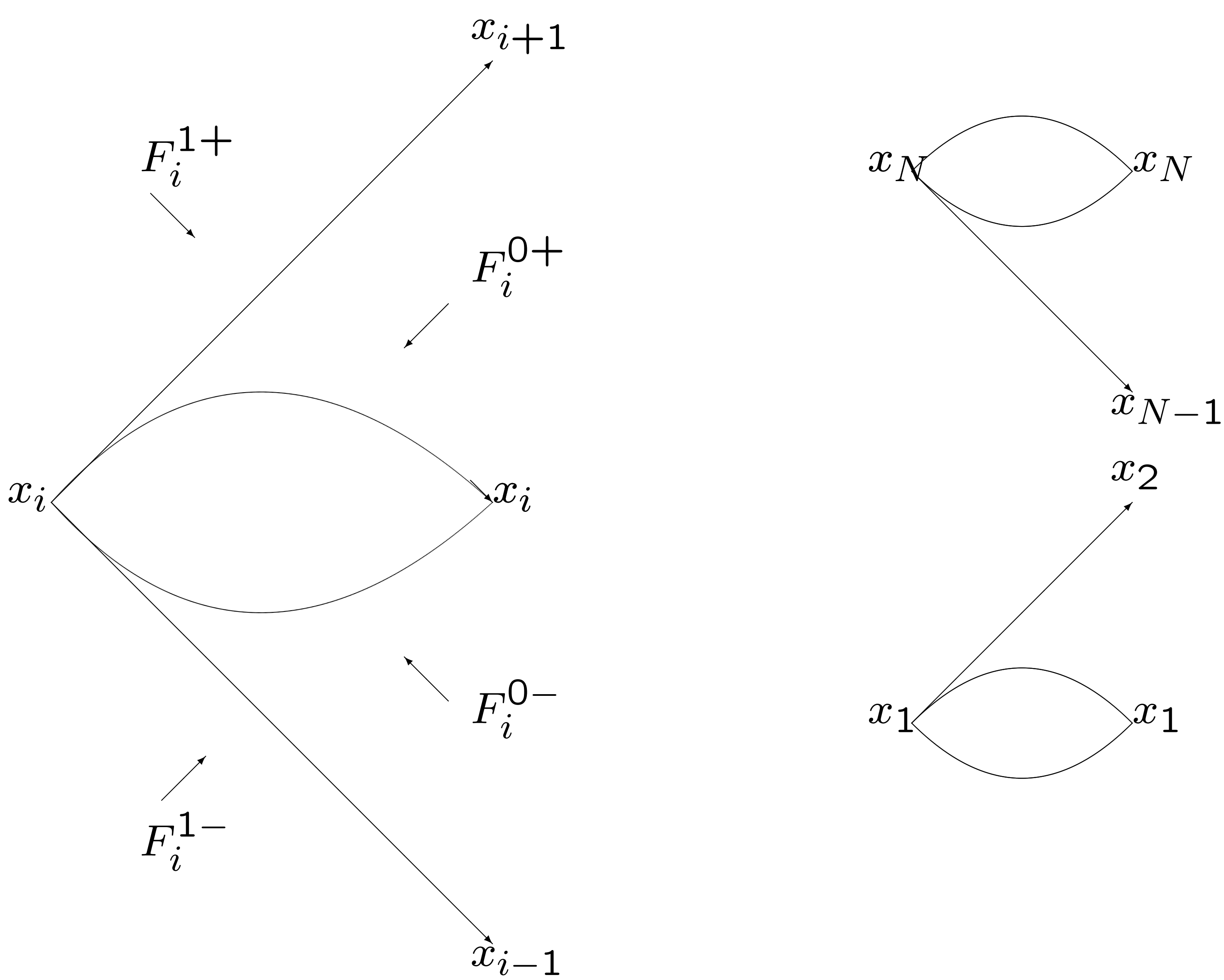
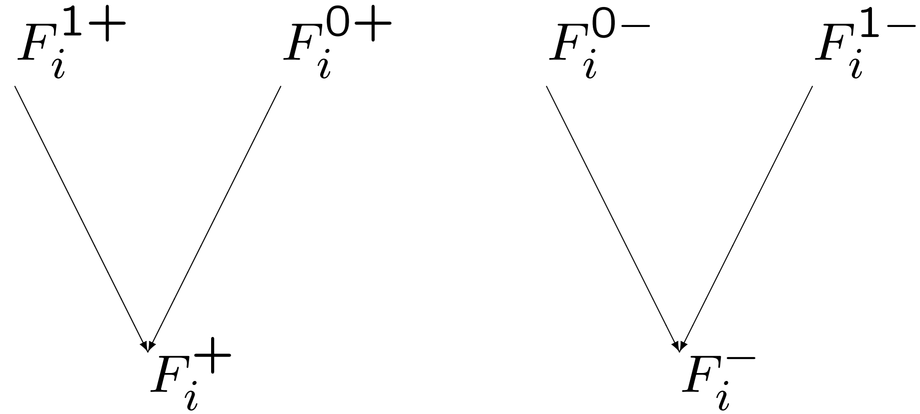


x_{i+1}
F_i^{1+}
F_i^{0+}
x_i
x_i
F_i^{0-}
F_i^{1-}
x_{i-1}
x_N
x_N
x_{N-1}
x_2
x_1
x_1


F_i^{1+}
F_i^{0+}
F_i^{0-}
F_i^{1-}
F_i^{+}
F_i^{-}

(a_n^x, b_n^x) the nth excursion interval whose length is greater than x.

- $a_n^x + x$ is a stopping time.

If the excursion during (a_n^x, b_n^x) started from x_i and is an upward excursion, then

$$P(b_n^x - a_n^x + x \in dy, X_{b_n^x} = x_{i+1} | \mathcal{F}_{a_n^x + x}) = \frac{F_i^{1+}(dy)}{F_i^{+}(x, \infty]} 1_{\{y>x\}}$$

and

$$P(b_n^x - a_n^x + x \in dy, X_{b_n^x} = x_i | \mathcal{F}_{a_n^x + x}) = \frac{F_i^{0+}(dy)}{F_i^{+}(x, \infty]} 1_{\{y>x\}}$$

Conditional hazard rate:

$$\lambda(t) = \begin{cases} 0 & \text{if } X_t \geq x_2 \\ \frac{f_2^{1-}(U_t)}{F_2^{-}[U_t,\infty)} & \text{if } x_1 < X_t < x_2 \end{cases}$$

Theorem

$$Y_t = \begin{cases} 1_{\{\tau \leq T\}} \quad \text{if } t \geq T \wedge \tau, \\ P^{x_i}(\tau \leq T - t) \quad \text{if } t < \tau \wedge T \text{ and } X_t = x_i \\ \boxed{\frac{1}{F_i^+[U_t,\infty]} \left(\int_{[U_t,\infty]} p_{i+1}(g_t + u) F_i^{1+}(du) + \int_{[U_t,\infty]} p_i(g_t + u) F_i^{0+}(du) \right)} \\ \quad \text{if } X \text{ is having an upward excursion from } x_i \\ \boxed{\frac{1}{F_i^-[U_t,\infty]} \left(\int_{[U_t,\infty]} p_{i-1}(g_t + u) F_i^{1-}(du) + \int_{[U_t,\infty]} p_i(g_t + u) F_i^{0-}(du) \right)} \\ \quad \text{if } X \text{ is having a downward excursion from } x_i, \end{cases}$$

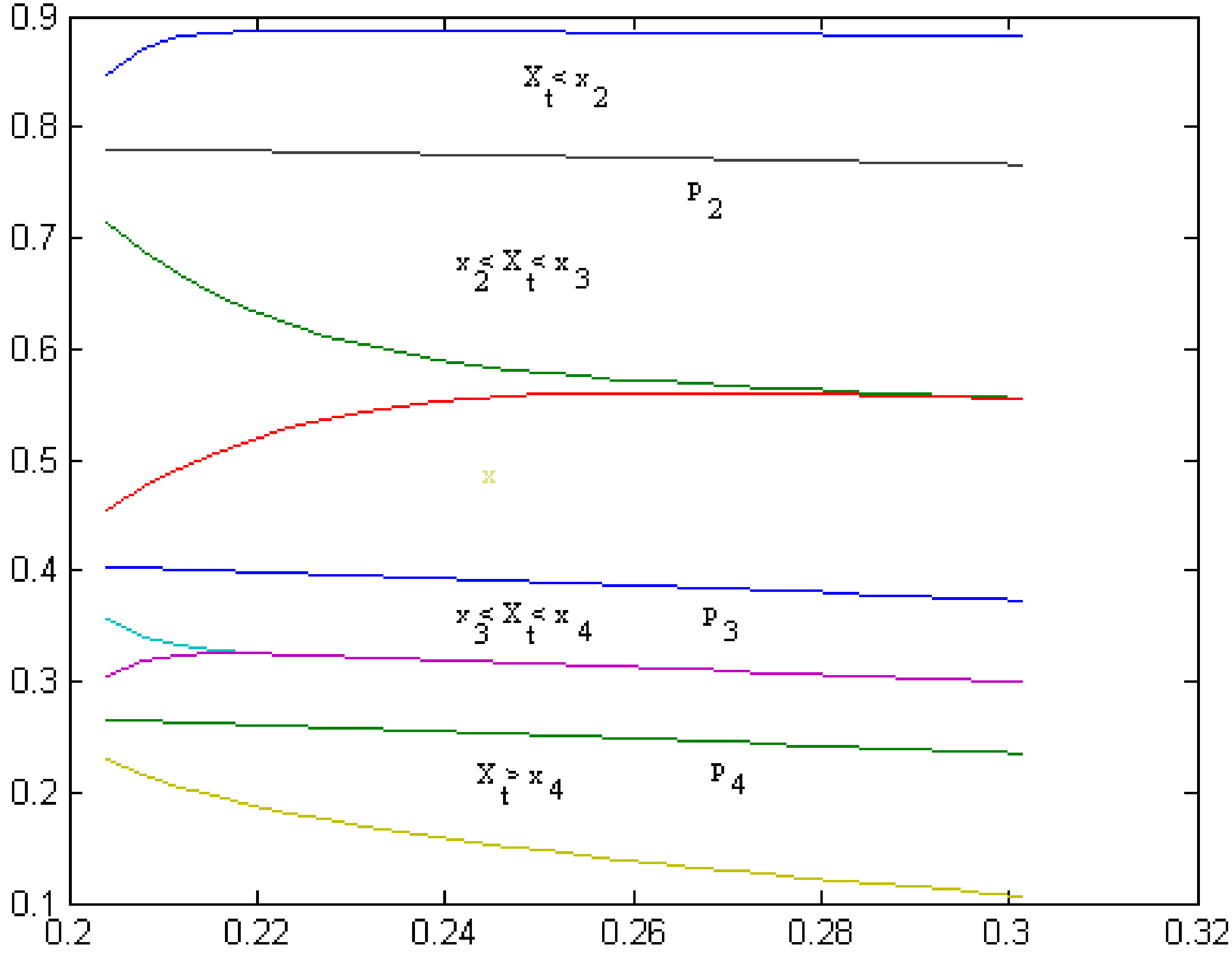
How do we compute $F_i^{j\pm}$?

• Laplace transforms $F_i^{j\pm}$ are expressed in terms of the solutions of $\mathcal{A}u = \lambda u$.

For qualitative results we consider the simplest case:

The coefficients of $\mathcal{A}$ are constant, i.e. X is Brownian motion with drift. i.e.,

$$\mathcal{A} = \mu \frac{d}{dx} + \frac{1}{2}\sigma \frac{d^2}{dx^2}$$



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0.2
0.22
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0.28
0.3
0.32
X t < x 2
P 2
x 2 < X t < x 3
x
x 3 < X t < x 4
P 3
X t > x 4
P 4

Jump rates:

$$\lambda_i^{1+}(x) = \frac{f_i^{1+}(x)}{F_i^{+}[x, \infty]}.$$

$$\lambda_i^{0+}(x) = \frac{f_i^{0+}(x)}{F_i^{+}[x, \infty]}.$$

Note

$$\lambda_t = \begin{cases} 0 \text{ if } X_t \geq x_2 \\ \lambda_2^{1-}(U_t) \text{ if } x_1 < X_t < x_2. \end{cases}$$

Theorem:

i) λ_i^{0+} is monotone decreasing with

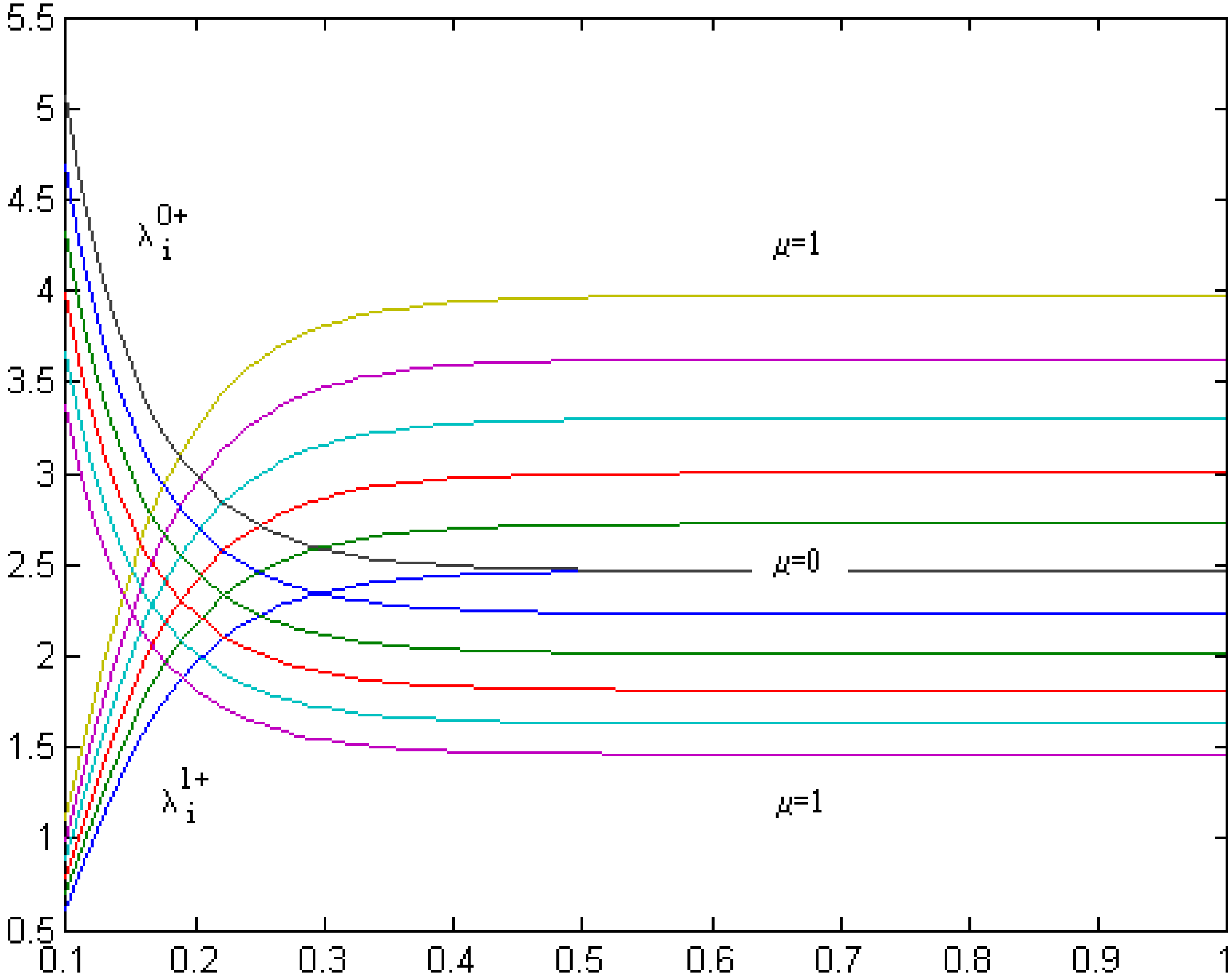
$$\lim_{x\to 0} \lambda_i^{0+}(x) = \infty,$$

$$\lim_{x\to\infty} \lambda_i^{0+}(x) = \left(\frac{\pi^2\sigma^2}{(x_i - x_{i+1})^2} + \frac{\mu^2}{2\sigma^2}\right)(1 + e^{\frac{\mu|x_i - x_{i+1}|}{\sigma^2}})^{-1}.$$

ii) λ_i^{1+} is monotone increasing with

$$\lim_{x\to 0} \lambda_i^{1+}(x) = 0,$$

$$\lim_{x\to\infty} \lambda_i^{1+}(x) = \left(\frac{\pi^2\sigma^2}{(x_i - x_{i+1})^2} + \frac{\mu^2}{2\sigma^2}\right)(1 + e^{-\frac{\mu|x_i - x_{i+1}|}{\sigma^2}})^{-1}.$$



5.5
5
4.5
4
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3
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2
1.5
1
0.5
0.1
0.2
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0.4
0.5
0.6
0.7
0.8
0.9
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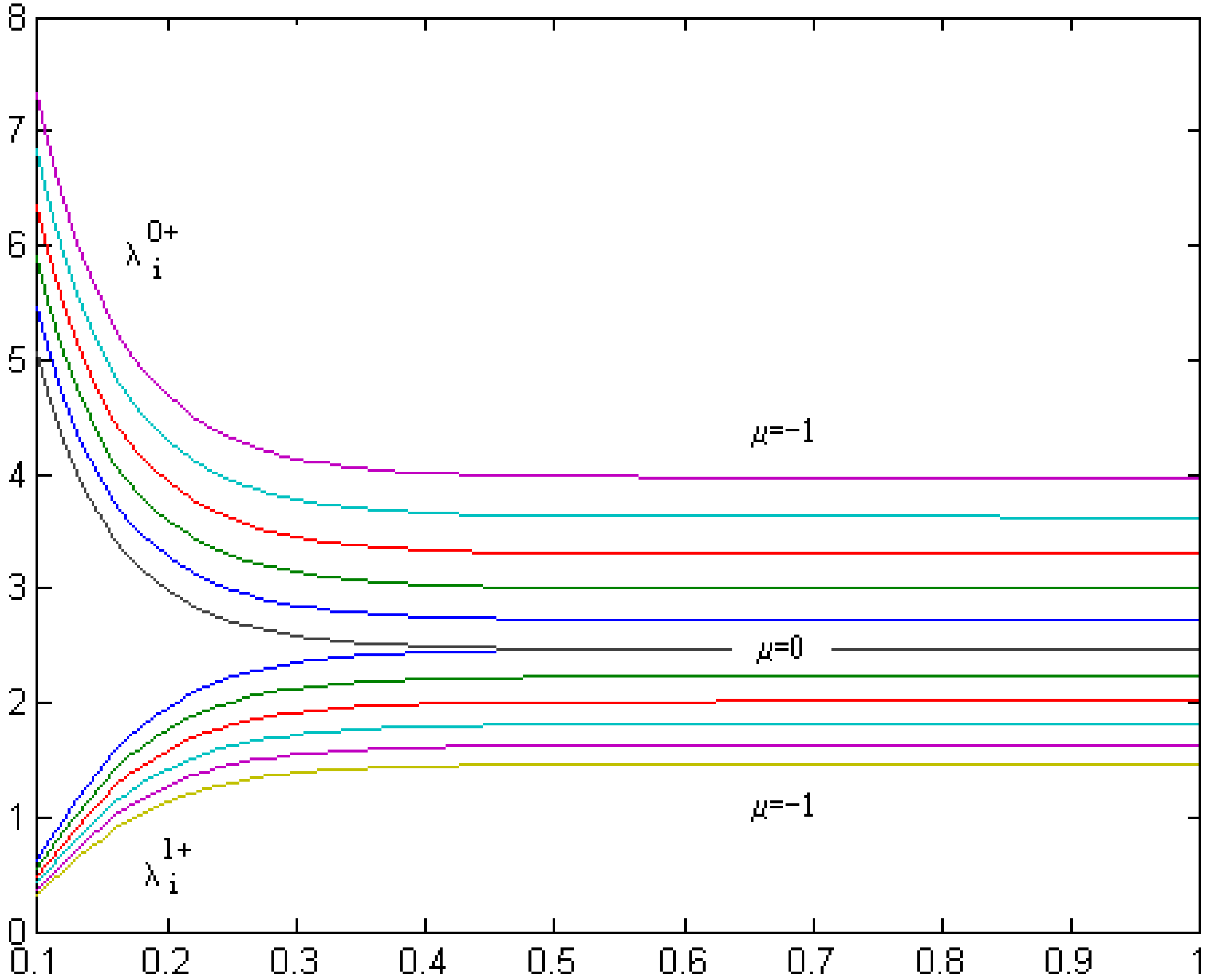
λ_i^{0+}
λ_i^{1+}
μ=1
μ=0
μ=1



8
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λ_i^{0+}
λ_i^{1+}
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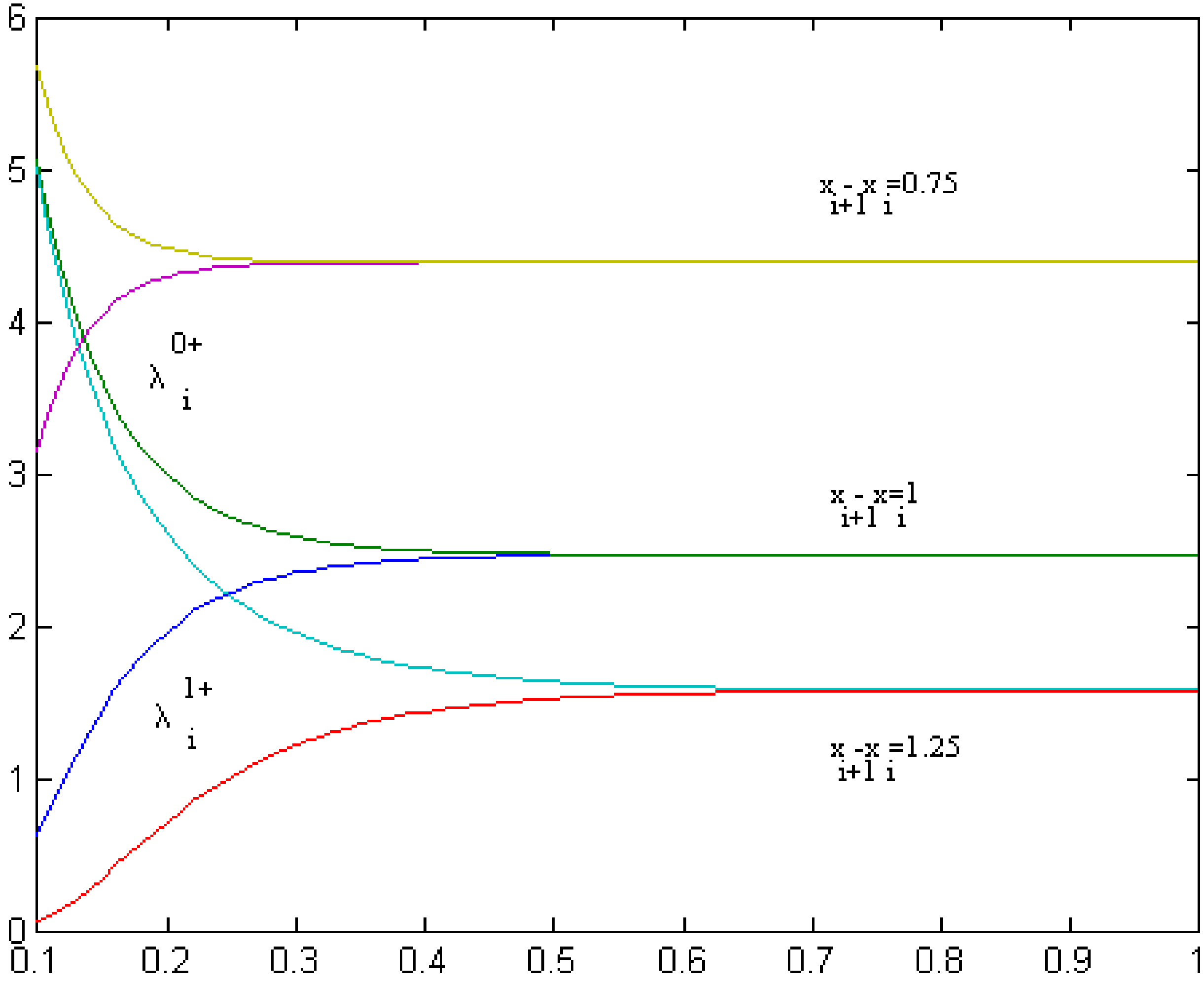
μ=-1



x - x =0.75
i+1 i
x - x=1
i+1 i
x -x =1.25
i+1 i
λ 0+
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λ 1+
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6
0.1
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1

Application Issues:

How to fit the market prices?

Following variables are needed:

$v(t_i, T_i), U_{t_i}, X_{g_{t_i}}, R(X_{t_i}), T_i - t_i$

How to choose the x_i?

These can be included in the model as parameters.

Open problem:

Equivalent martingale measure versus historical measure; how are these two related in a structural model with information reduction?