Chapter 6 - Modeling Default Risk

The Volatility Surface
A Practitioner's Guide
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Outline

Structural & Reduced models

firm based defaults & random defaults

Default risk model used by practitioners:

- `Structural` model - firm specific default
  - Occurs when some economic varaibles (firm value) crosses some barrier (debt)
  - Use a contingent claims model to find the probability of default.
  - H-W and Creditgrades are structural model.

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  - H-W and Creditgrades are structural model.
- `Reduced form` model - defaults as a random occurence
  - No latent variable which triggers the default event, just happens
  - They are easy to calibrate, but lack the ability to explain why default occurs
  - Duffie and Singleton (1999) and Merton (1970) are reduced form model.

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Merton's Model

Stock price jumping to zero

Merton's model of default

- Upon default, the stock price jumps to zero with $\lambda(t)$
- Jumps are independent of the stock price and $\mathbb{E}[J] = 0$.

To value a call option, because $V(SJ, t) = 0$ and using (5.3), we have

BS with shifted interest rate $r + \lambda$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial t} - rV - \lambda(t)\left\{V - S\frac{\partial V}{\partial S}\right\} = 0 \quad (1)$$

Shifted rate

Assuming no recovery

Assume no recovery (upon default) on the issuer's bonds. Then

- $B(JS, t) = 0$
- The risky bond price $B(t, T)$ must satisfy (1) with the solution

Risky bond price with shifted interest rate $r + \lambda$

$$B(t, T) = e^{-\int_t^T (r(s)+\lambda(s))ds}$$

- Shifted rate is the yield (= risk free rate+ credit spread) of a risky bond.

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Key Idea

BS formula could be a solution of an equation that has a jump to zero (*jump-to-ruin*) in it.

Volatility Skew

Skew that can become extremely steep for short dated options on stocks whose issuers have a higher credit spreads.

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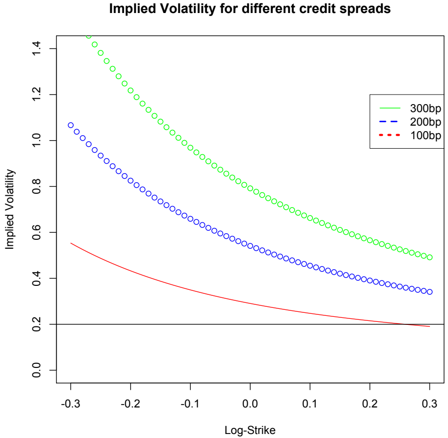

Capital Structure Arbitrage

Put-Call Parity

- Capital Structure arbitrage-arbitraging equity claims against fixed income and convertible claims (stocks, bonds and convertible bonds)
- the trader looks to see if equity puts are cheaper than credit derivatives and if so buys and sells the other
- Put-Call Parity

  Denoting the value of a risk- free put, call and bond by P_0, C_0, and B_0 and the value of risky claims on the issuer(I) of the stock by P_I, C_I, and B_I, we have

$$\begin{aligned} P_0 &= C_0 + KB_0 - S \\ &= C_I + KB_0 - S \\ &= P_I + S - KB_I + KB_0 - S \\ &= P_I + K(B_0 - B_I) \end{aligned}$$

Put-Call Parity

- risk-free put is worth more than the risky put
- the excess value is the difference in risky and risk-free bond prices (times the strike price).
- with maturity independent rates and credit spreads, $t = 0$,

$$B_0 - B_l = e^{-rT}\left(1 - e^{-\lambda T}\right),$$

  which is the discounted probability of default in the Merton model.
- the extra value is the strike price times the probability that default occurs.
- the payoff is also more or less exactly the payoff of a default put in the credit derivative market.

Local and Implied Volatility in Merton's model

- From (1.6), we have

$$\sigma_{\text{loc}}^2(K,S,T) = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}} \tag{2}$$

- Because BS for a call is linearly homogenous in S and K, we have $C = S\partial C/\partial S + K\partial C/\partial K$ and it follows that

$$K^2\frac{\partial^2 C}{\partial K^2} = S^2\frac{\partial^2 C}{\partial S^2}$$

- In the jump to ruin case with $r = 0, \delta = 0$, we have

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \lambda S\frac{\partial C}{\partial S} - \lambda C$$

- Writing in terms of K,

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2 K^2\frac{\partial^2 K}{\partial S^2} - \lambda K\frac{\partial C}{\partial K}$$

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Substituting the above results into (2) gives

$$\begin{aligned}\sigma^2_{\text{loc}}(K,S,T) &= \sigma^2 - \lambda \frac{K\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}} \\ &= \sigma^2 + 2\lambda\sigma\sqrt{T}\frac{N(d_2)}{N'(d_2)},\end{aligned}$$

with

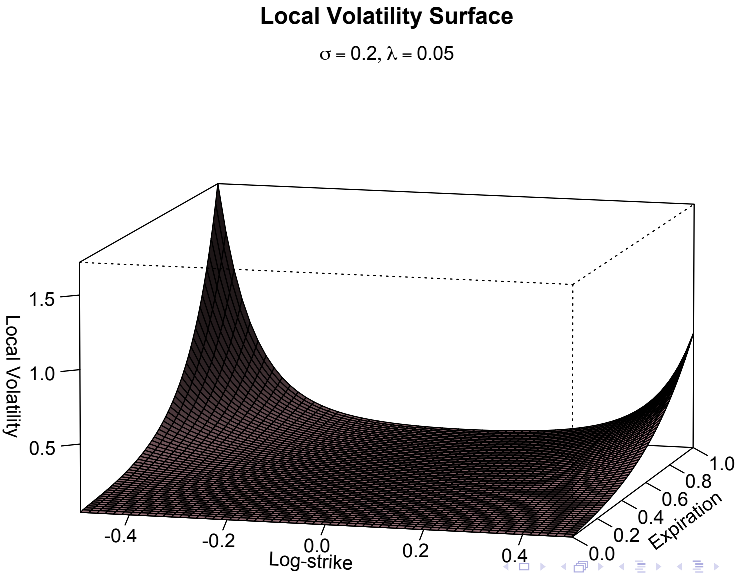
$$d_2 = \frac{\log S/K + \lambda T}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}.$$

For very low strikes $K/S << 1$, we have $d_2 >> 0$ and

$$N(d_2) \approx 1, \quad N'(d_2) = \frac{1}{\sqrt{2\pi}}e^{-d_2^2/2}$$

Local Volatility for very low strikes

$$\sigma^2_{\text{loc}}(K,S,T) \approx \sigma^2 + 2\lambda\sigma\sqrt{T}\sqrt{2\pi}e^{d_2^2/2} \qquad (3)$$


Local Volatility Surface
σ = 0.2, λ = 0.05
Local Volatility
1.5
1.0
0.5
Log-strike
-0.4
-0.2
0.0
0.2
0.4
Expiration
0.0
0.2
0.4
0.6
0.8
1.0

The Effect of Default Risk On Option Prices

Implied Volatilities of Jan. 2005 options on GT

Strike	Bid Vol	Bid Vol	Merton Vol
2.50		147.2%	145.2%
5.00	73.6%	88.3%	85.8%
7.50	48.3%	58.2%	51.2%
10.00	38.1%	45.0%	43.1%
12.50	41.2%	48.1%	41.5%
15.00	51.2%	54.3%	40.9%
17.50	64.5%	66.5%	40.6%
20.00		77.3%	40.0%
25.00		94.7%	40.0%
30.00		108.3%	40.0%

Table: Implied Volatility for January 2005 options on GT as of October 20, 2004 ($S = 9.4$)

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Fitting parameter $\lambda = 0.01934, \sigma = 0.3946$

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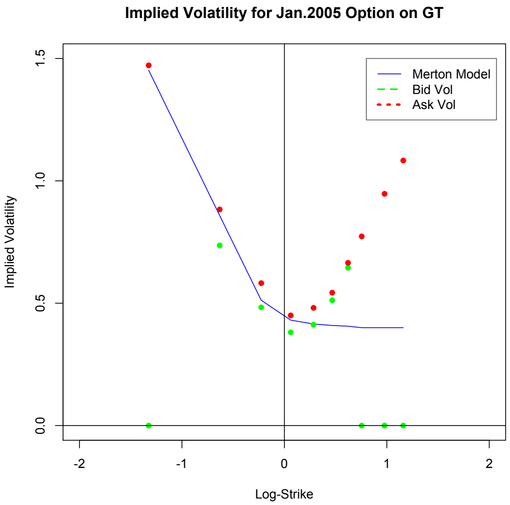

BS (1973) and Merton (1974)

- models equity as a call option on the value of a company
- The value of a company V is assumed to diffuse with no jumps.
- Debt is a call writer's position of long V and short the call on V
- As the Stock price declines, the company gets more leveraged and volatility increases.
- Generates an implied volatility skew too.

Big Practical Problem with the model

There is no way to generate significant short-dated credit spreads; default occurs when the value of the company hits a certain level below the current value and for short times, there is insufficient time for the V process to diffuse to the barrier.

Model Setup

V is assumed to evolve as a driftless geometric Brownian motion, so

$$\frac{dV_t}{V_t} = \sigma dW,$$

where σ is the volatility of firm value V.

- Level of V at which the company defaults is given by LD
  - D- today's value of its debt (per share)
  - L- recovery rate
- L-Lognormally distributed random variable with mean $\bar{L}$, and standard deviation λ, so that

$$LD = \bar{L}De^{\lambda Z - \lambda^2/2}, \quad Z \sim N(0,1)$$

- Z is assumed to be independent of W_t.

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Survival Probability

- Define

$$X_t = \sigma W_t - \lambda Z - \frac{\sigma^2 t}{2} - \frac{\lambda^2}{2}.$$

Then X_t is normally distributed with

$$\mathbb{E}[X_t] = -\frac{\sigma^2}{2}\left(t + \frac{\lambda^2}{\sigma^2}\right)$$

$$\mathrm{Var}[X_t] = \sigma^2\left(t + \frac{\lambda^2}{\sigma^2}\right)$$

- Default occurs when

$$V = V_0 e^{\sigma W_t - \sigma^2 t/2} = LD = \bar{L} D e^{\lambda Z - \lambda^2/2}$$

or equivalently when

$$X_t = \log\left(\frac{\bar{L}D}{V_0}\right) - \lambda^2$$

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Probability of Survival

Because X can be approximated by a Brownian motion with drift, the probability of survival (or the probability of not hitting the default barrier) is given by the Black Scholes like formula

$$P_t = N\left(-\frac{A_t}{2} + \frac{\log d}{A_t}\right) - dN\left(-\frac{A_t}{2} - \frac{\log d}{A_t}\right),$$

with

$$d = \frac{V_0 e^{\lambda^2}}{\bar{L}D}, \quad A_t^2 = \sigma^2 t + \lambda^2$$

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The Survival Probability

Since P_t is the probability of survival up to time t, it can be estimated directly from the prices of risky instruments such as bonds and Credit Default Swaps

The stock price S is approximately related (neglecting the time value of the option) to the firm value V via

$$V \approx LD + S$$

Then

$$\sigma \sim \frac{\delta V}{V} \approx \frac{\delta S}{S + LD} = \frac{\delta S}{S} \frac{S}{S + LD} \sim \sigma_S \frac{S}{S + LD}$$

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Stock Price and Volatility

Keeping σ fixed, as the stock price rises, the volatility σ_S of the stock declines. Conversly, as the stock price S approaches zero, the stock volatility increases as $1/S$.

In terms of market observables

$$P_t = N\left(-\frac{A_t}{2} + \frac{\log d}{A_t}\right) - dN\left(-\frac{A_t}{2} - \frac{\log d}{A_t}\right),$$

with

$$d = \frac{S_0 + \bar{L}D}{\bar{L}D} e^{\lambda^2}, \quad A_t^2 = \left(\sigma_S^* \frac{S^*}{S^* + \bar{L}D}\right)^2 t + \lambda^2$$

where S^* is some reference stock price and σ_S^* the stock volatility at that price.

Calibrating the model

Getting $\bar{L}, \lambda$ and D from the company and industry data rather than the term structure of the credit spreads would theoretically enables us to identify rich and cheap claims.

Summary

- Merton's jump-to-ruin model satisfies BS formula with interest rate shifted by hazard rate λ.
- Merton's model induces skew that can become extremely steep for short dated options on stocks whose issuers have high credit spreads
- Credit Spreads are explicitly related to the volatility skew