Fourier Transform in Finance - A Literature Review (First Draft)

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Abstract

This paper gives an overview on methodology and applications of Fourier Transform in Finance...

1 Introduction

A general definition of Fourier Transform (FT) can be given as follows.

Definition Let $f(t)$ be a function defined on $\mathbb{R}$. Given the Fourier Parameters (a, b), the Fourier Transform of f is defined as the function

$$\mathcal{F}(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(t) e^{ib\omega t} dt.$$

The domain of $\mathcal{F}$ consists of complex numbers ω that the improper integral converges.

One can obtain that the inverse Fourier transform is

$$f(t) = \sqrt{\frac{|b|}{(2\pi)^{1+a}}} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{-ib\omega t} d\omega.$$

Different choices of the Fourier Parameters (a, b) are used in different context. For example, $(0, 1)$ is used in modern physics, $(1, -1)$ is used in pure mathematics and systems engineering, $(1, 1)$ is used in classical physics, and $(0, -2\pi)$ is used in signal processing. In our review, the common choices of the papers are $(1, 1)$ and $(0, -2\pi)$.

The reader can find the standard properties of Fourier transforms such as time shift, convolution, etc. We are interested in computing Fourier transforms. When the transform does not have a closed form or is hard to obtain, the so called Discrete Fourier Transform (DFT) is used to approximate the FT in practice.

Definition Discrete Fourier Transform is a map from $\mathbb{C}^N$ to $\mathbb{C}^N$ that transform $(x_0, x_1, \cdots, x_{N-1}) \in \mathbb{C}^N$ to $(X_0, X_1, \cdots, X_{N-1})$ with

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i n k}{N}}.$$

Add exact relation between DFT and approximation of FT here. It is in Carr and Madan?????????

The so-called Fast Fourier Transform (FFT) refers to some efficient algorithms of computing DFT and its inverse. Computing DFT directly from the definition requires $O(N^2)$ arithmetical operations, but FFT algorithms can substantially improve the speed of the computation. The Cooley-Tukey algorithm, for instance, can compute DFT in $O(N \log N)$ operations.

Fourier transform methods have been one of the major methodologies for computing prices of financial derivatives in recent years. One of the reasons is that more sophisticated financial models are introduced beyond the classic Black-Schole model. These new models usually involve stochastic processes that are not normally distributed, which resulting difficulties in computing expectations. Fourier transform directly relates characteristic function to the corresponding density function, hence can be naturally implemented in the computation.

To illustrate how Fourier transform can be used we briefly introduce the original paper of Carr and Madan (1999).

Given an option with strike K, maturity T, and spot price S_t, let $k = \ln K$, $s_t = \ln S_t$, then the time zero price of the option can be considered as a function of k and is given by

$$C(k) = \int_k^\infty e^{-rT}(e^s - e^k) q(x) dx,$$

where $q(x)$ is the risk-neutral density function of s_T. One problem of computing the integral is that $q(x)$ usually does not have a closed form. Instead we usually have a closed form the the characteristic function $\phi(u)$. Then choose some $\alpha > 0$ such that $c(k) = e^{\alpha k} C(K)$ is square integrable and apply Fourier transform (with parameters $(1,1)$) to $c(k)$ to get

$$\psi(v) = \int_{-\infty}^\infty e^{ivk} c(k) dk.$$

Applying the inverse Fourier transform we can get

$$C(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^\infty e^{-ivk} \psi(v) dv \tag{1}$$

Using the fact that the characteristic function $\phi(u)$ of s_T is the Fourier transform of $q(x)$, one can show that $\psi(v)$ is linked to $\phi(u)$ by the equation

$$\psi(v) = \frac{e^{-rT}\phi(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \tag{2}$$

Combining (1) and (2) we can compute the option price. Note that (1) is a Fourier transform so FFT method can be used. To obtain $\psi(v)$ in equation (1) we only need the characteristic function $\phi(u)$.

Various application of FT and FFT are developed in recent year in different areas o financial derivatives, which are briefly reviewed in the Section 2. We classify the literature into five categories and each paper is introduced according to its category. We also include in Section 3 a time line of the development of the literature.

2 Classification of Literature on Fourier Transform Method in Finance

According to the content of the articles, we categorize the literature in the following five subsections.

2.1 Type of Derivatives

Anderluh and van der Weide (2009): on double-sided Parisian Options

The paper gives an overview of types of contracts that can be derived from the double-sided Parisian knock-in calls, discusses the Fourier inversion, and concludes with numerical examples to explain the behavior of the Parisian option. The paper also obtains a result [Corollary 3.3] on the probability that a standard Brownian motion makes an excursion of a given length below some level before it makes an excursion of another length above some other level.

Double-sided Parisian Options: The Parisian option is a kind of barrier option with the difference that the contract is not specified in terms of touching a barrier, but in terms of staying above or below the barrier for a certain period of time.

Deng (2000): on electricity derivatives

The paper proposes three mean-reversion jump-diffusion models (Section 2) to describe spot prices of electricity and derives the prices of various electricity derivatives under each model using Fourier Transform (Section 3).

Mean-reversion: a mathematical concept sometimes used for stock investing, but it can be applied to other assets. In general terms, the essence of the concept is the assumption that both a stock's high and low prices are temporary and that a stock's price will tend to move to the average price over time.

Benhamou (2002): on discrete Asian options

The paper presents a methodology for the discrete Asian options consistent with different types of underlying densities, especially non-normal returns. The method is an improved algorithm of the algorithm of Caverhill and Clewlow (1992) based on Fast Fourier Transform (FFT).

Discrete Asian options: Asian options are financial contracts giving the holder the right to buy a certain asset for a pay-off price related directly to its average price during a certain time period at a later date (expiry date). The averaging can be arithmetic or geometric, the average price can be either discretely sampled or continuous sampled.

Fusai (2004): on Asian options

The paper uses a Laplace transform with respect to time to maturity and a Fourier transform with respect to the logarithm of the strike to get a simple expression (see Theorem 1 on page 4) for a double transform of the option price of an Asian option. And then they use a multivariate version of the Fourier-Euler algorithm introduced in Abate and Whitt (1992) to obtain the option price. It turns out that the numerical inversion can get very accurate results, in particular for the difficult cases of low volatility levels.

2.2 New Methods

Carr and Madan (1999): One of the original papers which brought FFT to derivative evaluation. The main idea is described in Section 1. The detail is described in Section 1.

Raible (2000): In Chapter 3 of his Ph.D Thesis, Raible derived a formula of prices of European options by bilateral Laplace transform, and FFT method can be used to evaluate the formula.

Description of the method: For an European option on a stock with price $S_t = S_0 e^{X_t}$ and payoff function $g(S_T)$ at expiration T, the price of the option at time 0 is $V(c) = e^{-rT} E_Q[g(S_T)] = e^{-rT} E_Q[g(e^{-c+X_T})]$, where $c = -\ln S_0$ and Q is the risk-neutral probability. Let $\rho(x)$ be the density function of X_T.

Let $\pi(x) = g(e^{-x})$, then

$$V(c) = e^{-rT} E_Q[\pi(c - X_T)] = e^{-rT} \int_{\mathbb{R}} \pi(c - x)\rho(x)dx,$$

which is the convolution of π and ρ.

Let $\mathcal{L}_f(z)$ denoted the bilateral Laplace transform of $f(x)$ defined by $\mathcal{L}_f(z) = \int_{\mathbb{R}} e^{-zx} f(x)dx$, then using the property of Laplace transform we have $\mathcal{L}_V = e^{-rT}\mathcal{L}_\pi \mathcal{L}_\rho$. Then by taking the inverse bilateral Laplace transform we can get

$$V(c) = \frac{e^{-rT+cR}}{2\pi} \int_{\mathbb{R}} e^{icu} \mathcal{L}_\pi(R+iu)\mathcal{L}_\rho(R+iu)du,$$

where R is a real number so that both $\mathcal{L}_\pi(R+iu)$ and $\phi_{X(T)}(iR-u)$ are defined for $u \in \mathbb{R}$. But note that $\mathcal{L}_\rho(R+iu) = \phi_{X(T)}(iR-u)$ where $\phi_{X(T)}$ is the characteristic function of X_T, so we have

$$V(c) = \frac{e^{-rT+cR}}{2\pi} \int_{\mathbb{R}} e^{icu} \mathcal{L}_\pi(R+iu)\phi_{X(T)}(iR-u)du,$$

which can be considered as a Fourier Transform, hence FFT can be applied to its calculation.

Fang and Oosterlee (2008): the COS method

Description of the method: when computing the integral in the European style option price formulas, the method replaces the density function by it Fourier-cosine series expansion, and approximate the integral $\int_{\mathbb{R}}$ by $\int_a^b$ for sufficiently large interval $[a, b]$. Using the fact that a density function tends to be smooth and decay rapidly to zero, a few term of expansion may provide accurate approximation.

Jackson, Jaimungal, and Surkov (2007): a new efficient transform approach for regime-switching Levy models

Regime-switching Levy process: Regime-switching means that the world switches between states representing, for example, moderate, low and high volatilities regimes. The switching is continuous on time and is usually assumed to be a Markov Chain.

Description of the method: The algorithm begins with discretizing the continuous Markov chain into steps of Δt on which the Markov process remain constants. Then the algorithm uses the backward recursive formula

$$v^{n-1}(k) = FFT^{-1}\left[FFT[v^n(k)] \cdot e^{\Psi^{(k)}\Delta t}\right]$$

to get an efficient approximation of the price, where $\Psi^{(k)}$ is the Lévy exponent of the Lévy process in the k-th state.

Jaimungal and Surkov (2008): A new framework for dealing with mean-reverting jump-diffusion (and pure jump) models by working in Fourier space

Description of the method: The method is based on the Fourier space time stepping algorithm of Jackson, Jaimungal, and Surkov as we just described above, but is tailored for mean-reverting models.

Lord et al. (2008): The CONV Method-A fast and accurate method for pricing Bermudan options and American options

Bermudan option: A Bermudan option is an option where the buyer has the right to exercise at a set (always discretely spaced) number of times. This is intermediate between a European option an American option.

Description of the method: The paper reformulates the well-known risk-neutral valuation formula by considering it as a convolution. The resulting convolution is dealt with numerically by using the Fast Fourier Transform (FFT). For a Bermudan option, there is an backward recursive formula

$$C(t_m, x) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, x + z) f(z) dz,$$

The algorithm of pricing a Bermudan option is as follows.

1) $V(t_M, x) = E(t_M, x)$ for all x

2) For $m = M - 1$ to 0 Dampen $V(t_{m+1}, x)$ with $exp(\alpha x)$ and take its Fourier transform

Calculate C(tm, x) by applying Fourier inversion and undamping

$V(t_m, x) = \max E(t_m, x), C(t_m, x)$

3) Next m

American options can be priced using Richardson extrapolation on a series of Bermudan options with an increasing number of exercise opportunities.

Leentvaar and Oosterlee (2008): A Fourier-based sparse grid method for pricing multi-asset options

Description of the method: The paper proposes a parallel sparse grid method to compute option prices involving multiple assets such as call on geometric average of assets, basket call, and call on max or min of assets, in which cases the formula can be given by

$$V(t, \mathbf{x}) = e^{-r(T-t)} \int_{\mathbb{R}^d} V(T, \mathbf{x} + \mathbf{z}) f(\mathbf{z}) d\mathbf{z}$$

Computing the integral using FFT involves storage of the partition vector $N = \prod_{j=1}^{d} N_j$ which increases exponentially when d increases (referred as the curse of dimensionality). The parallel sparse grid method partition in a way that the j-th dimension partition is split into β_j parts where β_j is a power of 2. Then the FFT can be computed as sums of discrete Fourier transform

(DFT) with smaller sizes. The paper also includes complexity analysis and some computations illustrating the method.

Minenna and Verzella (2008): revisit a classic FT method

Description of the method: The paper revisits a classic FT method and change the quadrature algorithm to less flawed alternative, which can improve the computational performance up to levels that are only three time slower than FFT can achieve. This allows to have at the same time a reasonable computational speed and the well known stability and accuracy of canonical FT methods.

Wong and Guan (2010): a simple network approach to American exotic option valuation under Lévy processes using the fast Fourier transform (FFT).

Description of the method: The paper considers the transition probability $p_{ij} = P(S_{t+s} = s_j \mid S_s = s_i)$, $i, j = 1, 2, \cdots, N$. Let $X = \ln(S)$. Then the probability density function $f_t(y \mid x_i)$ of the increment $X_{t+s} - X_s$ conditional on $X_s = x_i$ is the inverse Fourier transform of characteristic function $\phi_t^X(z)$ for the increment $X_{t+s} - X_s$ conditional on $X_s = x_i$. Hence the FFT method can be applied to obtain an approximation of $f_t(y \mid x_i)$, which is then used to compute option price $C_N(s,t)$. The author shows that as N approaches infinity, $C_N(s,t)$ converges to the true value $C_E(s,t)$ for European options. By using backward iteration, similar result for American option prices is also obtained.

Chourdakis (2004): fractional FFT algorithm (FRFT)

Description of the method: The fractional FFT algorithm is developed by Bailey and Swarztrauber (1991), which can be used to rapidly compute sums of the form $\Sigma_{j=0}^{N-1} e^{-i2\pi kj\alpha} h_j$. In computing the fourier transform $\int_0^\infty e^{-ixu} h(u) du$, the advantage of using FRFT is that it allows the grid sizes δ, λ of u, x to be chosen independently. The paper compares the straight FFT of Carr and Madan (1999) and FRFT in computational speed and accuracy, and then conclude that FRFT can compute call options up to forty-five times faster than the straight fft of Carr and Madan (1999), without substantial losses in the accuracy of the results.

Ju and Zhong (2006): A method to compute the density of the arithmetic average of a Markov process. The method is then applied to price average rate options (i.e. Asian Options). It is applied to models in which the underlying price is following square-root process or log-normal process. Discussion in a more general setting is provided.

Description of the method: Let $x(t)$ be a nonnegative Markov process satisfying the PDE

$$dx(t) = \mu(x,t)dt + \sigma(x,t)dw(t)$$

where $w(t)$ is a standard Wiener Process. Let $y(t) = \int_0^t x(udu)$ be the arithmetic sum. The method makes use of the relation between the characteristic function F of $y(t)$ and its density function f. To determine F, the author shows that $F(\eta)$ satisfies the PDE

$$-i\eta xF + F_s + \mu(x,s)F_x + \frac{1}{2}\sigma^2 F_{xx} = 0,$$

which is the fundamental valuation equation for any contingent claim as if the interest rate were purely imaginary and the terminal condition resembles that of a default-free discount bond.

Liu, Zhang, and Yin (2006): An FFT method for the regime-switching model.

Description of the method: Assume under the risk-neutral probability $\widetilde{P}$ the underlying price follows the regime-switching model

$$\frac{dS(t)}{S(t)} = \mu(\alpha(t))dt + \sigma(\alpha(t))d\widetilde{B}(t)$$

where $\alpha(t)$ is a finite-state continuous time Markov chain with state space $M = \{1, 2, \cdots, m\}$. Then the paper applies the methodology of Carr and Madan (1999) and derives formulas similar to (1) and (2) in this specific setting. The cases when $m = 2$ and $m \geq 2$ are discussed. To reduce computational complexity, a near-optimal FFT scheme is proposed and tested when the modulating Markov chain has a large state space.

Attari (2004):

Description of the method: (To be edited) This paper presents a simple manipulation that (i) reduces the two numerical integrations needed to compute option prices using fourier inversion into a single numerical integration and (ii) reduces the number of characteristic function evaluations needed to obtain a given level of accuracy. The simplification does not preclude the use of other numerical techniques to increase computational speed. It will primarily help in empirical studies and in model calibration, since, these are occasions when computational efficiency is critical.

Chiarella, Carl, Ziogas, and Andrew (2009):

Description of the method: The paper considers the American option pricing problem in the case where the underlying asset follows a jump-diffusion process. The method of Jamshidian is applied to transform the problem of solving a homogeneous integro-partial differential equation (IPDE) on a restricted region to that of solving an inhomogeneous IPDE on an unrestricted region. Then Fourier transform technique is applied to the inhomogeneous IPDE to obtain the solution. New results concerning the limit for the free boundary at expiry are derived. The paper also present a numerical algorithm for the solution..

Hurd and Zhou (2009):
Description of the method: (To be edited) Spread options are a fundamental class of derivative contract written on multiple assets, and are widely used in a range of financial markets. There is a long history of approximation methods for computing such products, but as yet there is no preferred approach that is accurate, efficient and flexible enough to apply in general models. The present paper introduces a new formula for general spread option pricing based on Fourier analysis of the spread option payoff function. Our detailed investigation proves the effectiveness of a fast Fourier transform implementation of this formula for the computation of prices. It is found to be easy to implement, stable, efficient and applicable in a wide variety of asset pricing models.

2.3 Generalization of Methods

Andersen and Andreasen (2000): Generalize the implied diffusion approach of Dupire (1994) to asset processes with Poisson jumps and combined with Fast Fourier Transform methods.

Dempster and Hong (2000): The paper investigates a method for pricing the generic spread option by extending the fast Fourier Transform technique introduced by Carr and Madan (1999) from the classical two-factor Black-Scholes framework to a multi-factor setting. It is applicable when the the joint characteristic function of the underlying assets forming the spread is known analytically.

Zhylyevskyy (2009): a non-finite-difference-based method of American option pricing under stochastic volatility by extending the Geske-Johnson compound option scheme (add ref here)

Dufresne, Garrido, and Morales (2009): The paper shows Fourier methods may be used to compute excess-of-loss or stop-loss insurance. Their formulas require weaker assumptions. Numerical examples are provided.

2.4 General discussion

Eberlein, Glau, and Papapantoleon (2009): The paper provides a systematic analysis of the conditions required for the existence of Fourier transform valuation formulas in a general framework when the underlying variable can depend on the path of the price process and the payoff function can be discontinuous.

Cerny (2008): a concise introduction to applications of Fourier transform and FFT in option pricing

Cherubini, Lunga, Mulinacci, and Rossi (2010): a book containing systematic discussion and comprehensive background and development of FT Methods in Finance

Sepp (2003): The paper surveys the developments in the finance literature with respect to applying the Fourier transform for option pricing under affine jump-diffusions.

Schmelzle (2010): An introductory article including: review of the convenient mathematical properties of Fourier transforms and characteristic functions; survey on the most popular pricing algorithms and comparison of numerical quadratures for the evaluation of density functions and option prices; discussion on practical implementation details and possible refinements with respect to computational efficiency.

Lee (2004): a summary and unification of Fourier-analytic methods for pricing options on any underlying state variable whose characteristic function is known

Duffie, Pan, and Singleton (2000): In an economics view, the paper provides an analytical treatment of a class of transforms, including various Laplace and Fourier transforms, on problems include fixed-income pricing models as well as a wide range of option-pricing applications. In option pricing, the paper uses the affine stochastic volatility model proposed by Heston (1993).

2.5 Other

To be edited

Bormetti, Cazzola, Livan, Montagna, and Nicrosini (2009): risk

Albanese, Jackson, and Wiberg (2004): The paper introduces a new Fourier method for computing value-at-risk for a portfolio with derivatives and for return models with fat tails, which does not assume the characteristic function is known explicitly.

Gutierrez (2007): The paper investigates European options when the underlying is following a time-changed Brownian motion. The returns density function is expressed in a manner so the option prices can be transform directly without introducing dampening factors. The paper then obtains a simple expression for the option price. Although the expressions can be derived from Lee (2004) by making a particular choice of dampening factor, the paper shows the implicit dampening factor is a natural candidate, and makes the formulation more elegant than previous ones from a theoretical point of view.

Hilber, Reich, Schwab, and Winter (2009): comparison

Thulasiram and Thulasiraman (2003): The paper reports development of a multithreaded FFT pricing algorithm which is proposed in the proceeding Thulasiraman, Theobald, Khokhar, and Gao (2000) and performance evaluation on a multithreaded platform. The multithreaded FFT algorithm is based on the Cooley-Tukey algorithm and can be performed on multiprocessor computers. The results indicate that the FFT multithreaded algorithm for option pricing is efficient giving a relative speedup of 50% on 64 processors.

3 Time line of the literature

The following table summarizes a rough time line of the literature of Fourier Transform methods in finance. Note that it is not a comprehensive list of papers on the subject.

Paper	Description	Main References
Carr and Madan (1999)	The paper gives the original idea of using FFT to calculate option prices.	Bakshi and Madan (2000), Scott (1997)
Raible (2000)	In Chapter 3 of this Ph.D thesis, Raible derives a formula to calculate option prices by bilateral Laplace transform.	Carr and Madan (1999)
Duffie et al. (2000)	This paper extends the literature on affine asset-pricing models by deriving analytically tractable pricing relations for a wide variety of valuation problems.	Heston (1993), Bakshi and Madan (2000)
Dempster and Hong (2000)	The paper extend the fast Fourier Transform technique in Carr and Madan (1999) to a multi-factor setting for generic spread options.	Carr and Madan (1999), Bakshi and Madan (2000), Duffie et al. (2000)
Deng (2000)	The paper studies electricity derivatives and uses Fourier Transform to derive prices of various electricity derivatives.	Duffie et al. (2000), Heston (1993)
Benhamou (2002)	The paper studies discrete asian options and proposes an enhanced version of FFT algorithm of Andrew and Les (1992).	Carr and Madan (1999), Andrew and Les (1992)
Thulasiram and Thulasiraman (2003)	The paper reports the development of a multithreaded FFT pricing algorithm and performance evaluation on a multithreaded platform.	Carr and Madan (1999), Dempster and Hong (2000), Thulasiraman et al. (2000)
Lewis (2001)	ft wrt log spot price............	....
Chourdakis (2004)	The paper uses the method of fractional FFT to compute option prices and conducts experiments to show the efficiency of the method compared to straight FFT.	Bailey and Swarztrauber (1991), Carr and Madan (1999), Bakshi and Madan (2000)

Table Continued

Paper	Description	Main References
Fusai (2004)	The paper studies asian options. They use a double transform and a multivariate version of the Fourier-Euler inversion algorithm to obtain the option price.	Fu, Madan, and Wang (1998), Geman and Yor (1993), Abate and Whitt (1992)
Lee (2004)	The paper extends and unifies Fourier transform methods to price a wide class of options in a general setting. In particular, the paper examines the error control of discretized transform methods such as DFT/FFT.	Carr and Madan (1999), Duffie et al. (2000), Bakshi and Madan (2000), Lewis (2001)
Liu et al. (2006)	The paper first develops an FFT method for a model where the underlying price is governed by a regime-switching geometric Brownian motion. The method is tested and numerical results are reported.	Carr and Madan (1999), Yao, Zhang, and Zhou (2006)
Gutierrez (2007)	The paper finds a simple expression for European option prices when the underlying follows a time-changed Brownian motion. The method can avoid the use of external dampening factors.	Carr and Madan (1999), Lee (2004)
Jackson et al. (2007)	....	....
Lord et al. (2008)	The paper propose a method called CONV which recognize the well-known risk-neutral valuation formula as a convolution and then evaluate it by FFT.	Andricopoulos et al. (2003), OSullivan (2005)
Leentvaar and Oosterlee (2008)	The paper proposes a parallel partitioning method for computing price of multi-asset options when evaluating FFT.	Carr and Madan (1999), Lord et al. (2008), Raible (2000)

Table Continued

Paper	Description	Main References
Fang and Oosterlee (2008)	The paper propose a option pricing method called COS which uses the relation of the characteristic function with the series coefficients of the Fourier-cosine expansion of the density function.	Lord et al. (2008)
Minenna and Verzella (2008)	In this paper the canonical method in Heston (1993) is modified with a Gauss-Lobatto quadrature scheme. It can improve the computational speed up to levels that are only three time slower than FFT and keep the stability and accuracy of canonical FT methods.	Heston (1993), Gander and Gautschi (2000)
Chiarella et al. (2009)	The paper studies American option pricing problem where the underlying asset follows a jump-diffusion process. Fourier transform technique is applied to an inhomogeneous IPDE to obtain the solution in the form of a pair of linked integral equation. The paper also present a numerical algorithm for the solution.	Jamshidian (2007), Kallast and Kivinukk (2003)
Dufresne et al. (2009)	The paper shows Fourier methods may be used to compute excess-of-loss or stop-loss insurance.	Carr and Madan (1999), Raible (2000)
Eberlein et al. (2009)	The paper provides a systematic analysis of the conditions such that Fourier transform valuation formulas are valid in a general framework.	

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