The Volatility Surface (Jim Gatheral)

Chapter 11: Volatility Derivatives

Tony Ware
University of Calgary

10th March, 2011

Main results from last week…

PDF of stock price at T, given S_t at t:

$$p(K; T, t, S_t) = \frac{\partial^2}{\partial K^2} \widetilde{C}(S_t, K, t, T) = \frac{\partial^2}{\partial K^2} \widetilde{P}(S_t, K, t, T).$$

Prices of European claims:

$$\mathbb{E}[g(S_T)|S_t] = g(F) + \int_0^F \widetilde{P}(k) g''(k) dk + \int_F^\infty \widetilde{C}(k) g''(k) dk,$$

where $F = \mathbb{E}[S_T|S_t]$.

Three examples

$$g(x) = (x - L)_+, \qquad g(x) = \frac{(x - L)_+}{x}, \qquad g(x) = \log \frac{x}{F}.$$

Main results from last week...

Pricing variance swaps

For diffusions...

$$\mathbb{E}\left[\int_0^T \sigma^2(S_t)dt\right] = -2\mathbb{E}\left[\log\frac{S_T}{F}\right] = 2\int_{-\infty}^0 p(k)dk + 2\int_0^\infty c(k)dk,$$

where $c(y) = \frac{\widetilde{C}(Fe^y)}{Fe^y}$ and $p(y) = \frac{\widetilde{P}(Fe^y)}{Fe^y}$, giving us a model-independent formula for the price of a variance swap.

The Heston model

$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t}dZ_t.$$

Let $\hat{v}(t) := \mathbb{E}[v_t|v_0]$. Then $\hat{v}(t) = \bar{v} + e^{-\lambda t}(v_0 - \bar{v})$, and so $\frac{1}{T}\mathbb{E}[W_T]$, the exected annualized variance up to T, is $\bar{v} + (v_0 - \bar{v})\frac{(1 - e^{-\lambda T})}{\lambda}$.

Dependence of $\mathbb{E}[W_T]$ on implied volatility skew

Claim: $\frac{1}{T}\mathbb{E}[W_T] = \int_{\mathbb{R}} N'(z)\sigma^2_{\mathrm{BS}}(z)dz$—so that it depends only on the *even* component.

Notation: $k := \log K/F$, $\lambda(k) := \sigma_{\mathrm{BS}}(k)\sqrt{T}$, $z(k) := d_2 = -\frac{k}{\lambda(k)} - \frac{\lambda(k)}{2}$.

$$\begin{aligned}\mathbb{E}[W_T] = -2\mathbb{E}[\log S_T/F] &= -2\int_0^\infty \log\frac{K}{F}\frac{\partial^2 \widetilde{C}}{\partial K^2}dK \\ &= -2\int_{-\infty}^\infty \frac{k}{K}\left(\frac{\partial^2 \widetilde{C}}{\partial k^2} - \frac{\partial \widetilde{C}}{\partial k}\right)dk.\end{aligned}$$

But $\widetilde{C} = F\{N(z+\lambda) - \mathrm{e}^k N(z)\}$, so $\frac{\partial \widetilde{C}}{\partial k} = F\mathrm{e}^k\{N'(z)\lambda' - N(z)\}$, and then $\frac{\partial^2 \widetilde{C}}{\partial k^2} - \frac{\partial \widetilde{C}}{\partial k} = F\mathrm{e}^k N'(z)\{\lambda'' - z' - \lambda' z z'\}$.

$$\begin{aligned}\mathbb{E}[W_T] &= 2\int_{-\infty}^\infty N'(z)k\{z' + \lambda' z z' - \lambda''\}dk = 2\int_{-\infty}^\infty N'(z)k\{\lambda' + kz'\}dk \\ &= -\int_{-\infty}^\infty N'(z)\lambda^2(k)z'dk = \int_{-\infty}^\infty N'(z)\sigma^2_{\mathrm{BS}}(z)Tdz. \leftarrow\end{aligned}$$

Is this correct?

The effect of jumps

If x_t denotes the return of a compound Poisson process with arrival rate λ: $x_T = \sum_i^{N_T} y_i$, then the quadratic variation is $\langle x \rangle_T = \sum_i^{N_T} |y_i|^2$, and

$$\mathbb{E}[\langle x \rangle_T] = \lambda T \int_0^\infty y^2 \mu(y) dy = \mathrm{Var}[x_T].$$

We can compute this in terms of call and put prices. But, if the process is a diffusion, $\mathbb{E}[\langle x \rangle_T] = -2\mathbb{E}[x_T]$.

If we do not know what the process is, and we use this formula, what kind of error are we committing?

Using the Lévy-Khintchine representation of the risk-neutral process,

$$\mathbb{E}[\langle x \rangle_T] + 2\mathbb{E}[x_T] = 2\lambda T \int_0^\infty \left(1 + y + \frac{y^2}{2} - e^y\right) \mu(y) dy.$$

This will be of the order of $\mathbb{E}[y^3]$, which may not be too bad.

Volatility swaps

Realized volatility is the square root of the realized variance; there is no static replicating portfolio available, and there will be a (model-dependent) *convexity adjustment*.

Convexity adjustment in the Heston model

$$\mathbb{E}\left[\sqrt{W_T}\right] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - \mathbb{E}\left[e^{-\psi W_T}\right]}{\psi^{\frac{3}{2}}} d\psi.$$

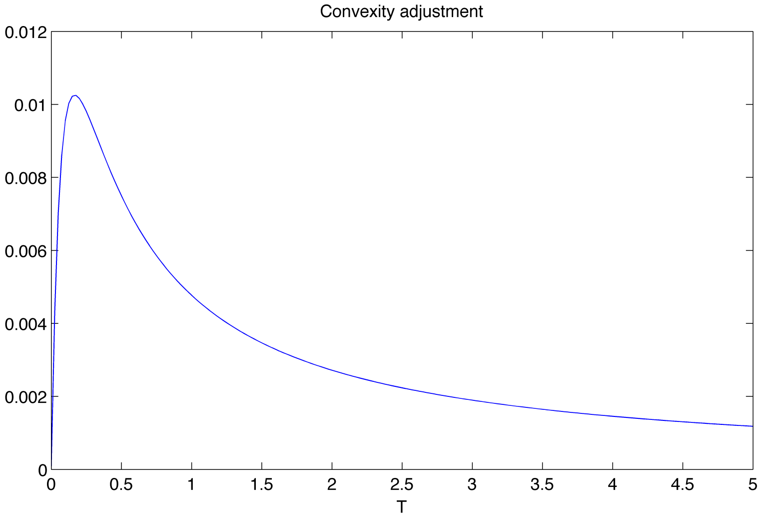
The right hand side can be found from the formula for the value of a bond in the CIR model (Cox *et al.* (1985)): $\mathbb{E}\left[e^{-\psi W_T}\right] = A e^{-\psi v_0 B}$, with

$$A = \left\{ \frac{2\phi e^{(\phi+\lambda)\frac{T}{2}}}{(\phi+\lambda)(e^{\phi T}-1)+2\phi} \right\}^{\frac{2\lambda\bar{v}}{\eta^2}} \text{ and } B = \frac{2(e^{\phi T}-1)}{(\phi+\lambda)(e^{\phi T}-1)+2\phi},$$

and $\phi = \sqrt{\lambda^2 + 2\psi\eta^2}$. The integral must be computed numerically.

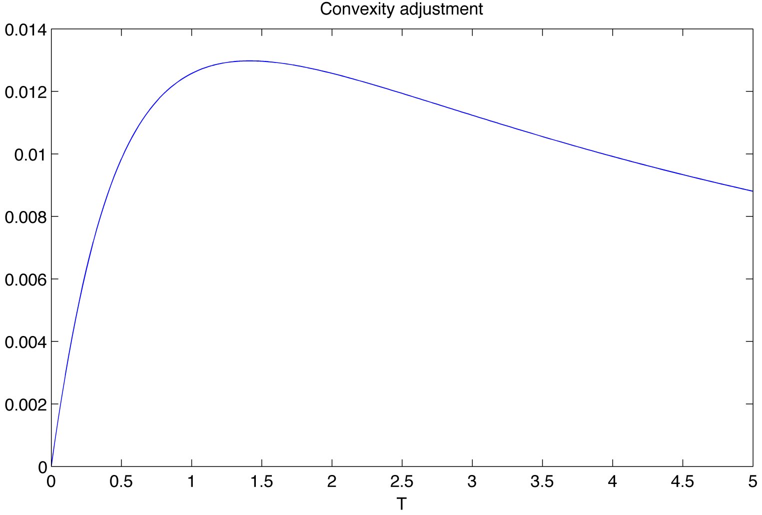
Volatility swaps

Convexity adjustment in the Heston model: $v_0 = 0.04$, $\bar{v} = 0.04$, $\lambda = 10.0$, $\eta = 1.0$



Volatility swaps

Convexity adjustment in the Heston model: $v_0 = 0.04$, $\bar{v} = 0.04$, $\lambda = 1.15$, $\eta = 0.39$



Valuing volatility derivatives

Assuming zero correlation between moves in the underlying and in volatility, any derivative with payoff in terms of $\langle x \rangle_T$ can be valued using calls and puts (Carr & Lee (2003); Friz & Gatheral (2005)).

The power payoff

$$\begin{aligned}\mathbb{E}[S_T^p] &= F^p \left\{1 + p(p-1)\left[\int_{-\infty}^{0} p(k)\mathrm{e}^{pk}dk + \int_0^\infty c(k)\mathrm{e}^{pk}dk\right]\right\} \\ &= F^p \left\{1 + 2p(p-1)\int_0^\infty c(k)\mathrm{e}^{k/2}\cosh\left[\left(p-\frac{1}{2}\right)k\right]dk\right\}.\end{aligned}$$

Quadratic variation under zero correlation

$$\mathbb{E}\left[\mathrm{e}^{\lambda\langle x\rangle_T}\right] = 1 + 4\lambda\int_0^\infty c(k)\mathrm{e}^{k/2}\cosh\left[\left(p(\lambda)-\frac{1}{2}\right)k\right]dk,$$

with $p(\lambda) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}$.

Volatility under zero correlation

Although the previous result would seem to indicate that we can write the fair value of any European-style claim on realized variance in terms of a portfolio of calls and puts, in general the coefficients in that portfolio are not well-defined.

$$\mathbb{E}[\sqrt{\langle x \rangle_T}] = \sqrt{2\pi} c(0) + \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{e}^{k/2} c(k) I_1 \left(\frac{k}{2}\right) dk,$$

where $I_n(\cdot)$ denotes a modified Bessel function of the first kind.

The coefficient of $c(0)$ thus has a delta-distribution weight, and dominates all the rest of the coefficients entirely. For example, in the case of a flat one-year volatility smile $\sigma_{\mathrm{BS}}(k, 1) \equiv 0.2$, the first term evaluates to 0.1977.

Replication will thus involve maintaining a position in the at-the-money option, which changes from day to day.

A simple lognormal model

Assume that $\log \sqrt{\langle x \rangle_T}$ is normally-distributed with mean μ and variance s^2. Then $\mathbb{E}[\sqrt{\langle x \rangle_T}] = e^{\mu + s^2/2}$ and $\mathbb{E}[\langle x \rangle_T] = e^{2\mu + 2s^2}$. The convexity adjustment is then

$$\sqrt{\mathbb{E}[\langle x \rangle_T]} - \mathbb{E}[\sqrt{\langle x \rangle_T}] = \left(e^{s^2/2} - 1\right) \mathbb{E}[\sqrt{\langle x \rangle_T}].$$

In this model (which we recall is surprisingly accurate), these values are all that is required to fix values of all options on variance.

Comparison with Heston for a variance call:

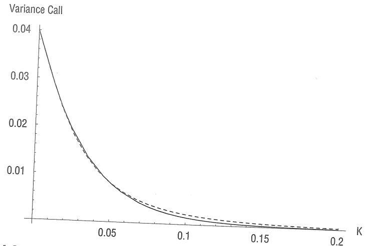


FIGURE 11.4 Value of 1-year variance call versus variance strike K with the BCC parameters. The solid line is a numerical Heston solution; the dashed line comes from our lognormal approximation.

Listed quadratic variation-based securities

The VIX Index

4. Why is the CBOE making changes to the VIX?

CBOE is changing VIX to provide a more precise and robust measure of expected market volatility and to create a viable underlying index for tradable volatility products.

- The New VIX calculation reflects the way financial theorists, risk managers and volatility traders think about - and trade - volatility. As such, the New VIX calculation more closely conforms to industry practice than the original VIX methodology. It is simpler, yet it yields a more robust measure of expected volatility. The New VIX is more robust because it pools information from option prices over a wide range of strike prices thereby capturing the whole volatility skew, rather than just the volatility implied by at-the-money options. The New VIX is simpler because it uses a formula that derives the market expectation of volatility directly from index option prices rather than an algorithm that involves backing implied volatilities out of an option-pricing model.
- The changes also increase the practical appeal of VIX. As noted previously, the New VIX is calculated using options on the S&P 500 index, the widely recognized benchmark for U.S. equities, and the reference point for the performance of many stock funds, with over $800 billion in indexed assets. In addition, the S&P 500 is the domestic index most often used in over-the-counter volatility trading.
- This powerful calculation supplies a script for replicating the New VIX with a static portfolio of S&P 500 options. This critical fact lays the foundation for tradable products based on the New VIX, critical because it facilitates hedging and arbitrage of VIX derivatives. CBOE has announced plans to list VIX futures and options in Q4 2003, pending regulatory approval. These will be the first of an entire family of volatility products.

Listed quadratic variation-based securities

The VIX Index

$$\text{VIX}^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} Q(K_i) - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2,$$

where $Q(K_i)$ is the price of the out-of-the-money option with strike K_i, and K_0 is the highest strike below the forward price F.

From our formulae:

$$\begin{aligned} \frac{\text{VIX}^2 T}{2} &= \int_0^\infty \frac{Q(K)}{K^2} dK + \int_{K_0}^F \frac{K-F}{K^2} dK \\ &\approx \int_0^\infty \frac{Q(K)}{K^2} dK - \frac{(K_0 - F)^2}{2K_0^2}. \end{aligned}$$

The formula above is one possible discretization for this.

The chapter ends with a brief discussion of Dupire's method for valuing VBX futures.

References

Carr, P., & Lee, R. 2003. Robust replication of volatility derivatives. *Unpublished paper: Courant Institute, NYU*.

Cox, J.C., Ingersoll Jr, J.E., & Ross, S.A. 1985. A theory of the term structure of interest rates. *Econometrica: Journal of the Econometric Society*, **53**(2), 385–407.

Friz, P., & Gatheral, J. 2005. Valuation of volatility derivatives as an inverse problem. *Quantitative Finance*, **5**(6), 531–542.