

Modern Portfolio Theory

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Lunch at the Lab, March 17 2011

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Market Model

We call a market the tuple

$\mathcal{M} := \left((\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}_{t \in [0, T]}, S(t), T \right)$, where (Ω, \mathcal{F}, P) is a probability space with filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$, $S_i(t), i = 1, \dots, m$, are the assets' prices, $S_0(t)$ is the bond price, T is the time horizon.

- ▶ $N(t) = (N_0(t), N_1(t), \dots, N_m(t))'$ is a **trading strategy** if $N_i(t)$ shares are held in the asset $S_i(t)$. The **wealth process** is

$$X(t) = \sum_{i=0}^m N_i(t) S_i(t).$$

- ▶ $\pi(t) := (\pi_1(t), \dots, \pi_m(t))' \in R^m$ is the **portfolio** if $\pi_i(t) = \frac{N_i(t) S_i(t)}{X(t)}$, $i = 1, \dots, m$, $\pi_0(t) = 1 - \pi(t) \mathbf{1}$.

- ▶ The trading strategy is **self-financing** if

$$dX(t) = \sum_{i=0}^m N_i(t) dS_i(t).$$

Market Assumptions 1

- ▶ The market is *frictionless*.
- ▶ The investor is a *small investor*, i.e. his actions do not affect the prices.
- ▶ Investors are rational.
- ▶ The market is efficient.

We say that a market is **frictionless** if the following conditions are satisfied

- 1) The securities are perfectly divisible.
- 2) Negative positions in securities are possible.
- 3) Rebalancing of the holdings does not lead to transaction costs.

An investor is **rational** if, given two portfolios that offer the same expected return, the investor will prefer the less risky one.

Efficient Market Hypothesis

Weak EMH: prices on traded assets already reflect all past publicly available information. **Semi-strong EMH:** prices reflect all publicly available information and they instantly change to reflect new public information. **Strong EMH:** prices instantly reflect even hidden or "insider" information.

There is evidence for and against the weak and semi-strong EMHs, while there is powerful evidence against strong EMH.

Sudden market crashes as happened on Black Monday in 1987 are mysterious from the perspective of efficient markets, but allowed as a rare statistical event under the Weak-form of EMH.

Individual investors trade too much, maintain undiversified portfolios, hold losing positions too long, require a risk premium for idiosyncratic risk, and overinvest in their own companies' stock. See [3].

Discrete Time Portfolio Optimization

Portfolio Optimization Problem

The portfolio optimization problem is a problem of a single agent who either aims to reduce the risk of his portfolio by rebalancing his positions in the assets, while keeping the targeted level of wealth, or aims to maximize his utility of wealth and/or consumption.

Markowitz portfolio problem

- ▶ Pioneered by H. Markowitz, a Nobel Prize Winner in Economic Sciences in 1990, together with Merton H. Miller, William F. Sharpe
- ▶ Single time-period, simplifies reality.
- ▶ Mathematical formulation of diversification; an important advance in the mathematical modeling of finance.

Market Assumptions 2

1. The investor makes one decision at $t = 0$.
2. The securities prices are modeled via their returns $R_i := \frac{S_i(T)}{S_i(0)}$, $i = 1, \dots, m$, with

$$b_i = E[R_i], \quad \sigma_{ij} = \text{Cov}(R_i, R_j), \quad \Gamma = (\sigma_{ij})_{m \times m}$$

3. The bond has a fixed rate of return r .

The choice of the portfolio π leads to a total return R equal to

$$R = \sum_{i=1}^m R_i \pi_i + r \pi_0.$$

The expected value and the variance of the portfolio return are

$$\begin{aligned} \mu(\pi) &= \sum_{i=1}^m b_i \pi_i + r \pi_0 = (b - r\mathbf{1})' \pi + r \\ \sigma^2(\pi) &= \pi' \Gamma \pi \end{aligned}$$

Problem formulation

For a given targeted level of the expected return μ_0 , choose an admissible portfolio π such that $\sigma^2(\pi)$ is minimal under all *admissible portfolios* π , i.e. portfolios that satisfy $(b - r\mathbf{1})' \pi = \mu_0 - r$, i.e.

$$\begin{aligned} & \min_{\pi \in R^m} \sigma^2(\pi) \\ & \text{subject to } (b - r\mathbf{1})' \pi = \mu_0 - r. \end{aligned}$$

This is a quadratic programming problem. If Γ is positive definite, then the problem possesses a unique solution

$$\begin{aligned} \pi^* &= \gamma \Gamma^{-1} (b - r\mathbf{1}) \\ \pi_0^* &= 1 - \mathbf{1}' \pi^*, \end{aligned}$$

where

$$\gamma = \frac{\mu_0 - r}{(b - r\mathbf{1})' \Gamma^{-1} (b - r\mathbf{1})}.$$

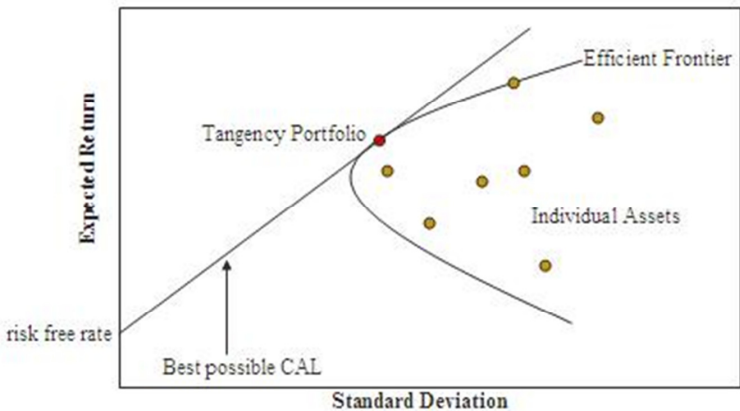
- ▶ π^* is **mean-variance efficient**.
- ▶ The set of all mean-variance efficient portfolios is called **the efficient set**.
- ▶ The set of ordered pairs $(\mu(\pi^*), \sigma^2(\pi^*))$, is **the efficient frontier**.

If we define

$$\pi_T = (0, \pi_1^*, \dots, \pi_m^*) \text{ and } \pi_g = (\pi_0^*, 0, \dots, 0),$$

then all minimum-variance efficient portfolios are linear combinations of π_T and π_g .

See [10].



On the efficient frontier,

$$\sigma^2(\pi) = \frac{(\mu_0 - r)^2}{[(b - r\mathbf{1})' \Gamma^{-1} (b - r\mathbf{1})]^2}$$

Continuous Time Portfolio Optimization

Market Assumptions 3

1. $m + 1$ assets are traded continuously on a finite horizon $[0, T]$.
2. m of these assets are stocks that follow the generalized Black-Scholes dynamics

$$dS_i(t) = S_i(t) \left(b_i(t) dt + \sum_{j=1}^m \sigma_{ij}(t) dW^j(t) \right), \quad t \in [0, T], \quad S_i(0) > 0,$$

where $b(t) := (b_1(t), \dots, b_m(t))'$ is the vector of stocks' drifts, and $\sigma(t) := (\sigma_{ij}(t))$ is the volatility matrix.

3. One of the assets is a bond, with the price $S_0(t)$,

$$dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T], \quad S_0(0) = S_0 > 0,$$

4. The number of traded stocks equals the number of sources of randomness which are m independent Wiener processes.
5. $\sigma(t)$, $\sigma^{-1}(t)$, $b(t)$ and $r(t)$ are \mathcal{F}_t measurable, bounded processes.
6. $\sigma(t)$ satisfies the non-degeneracy condition: for some $\delta > 0$,

$$x' \sigma(t) \sigma(t)' x \geq \delta x' x, \quad \forall t \in [0, T], \quad \forall x \in R^m.$$

Definition

$B(t) := b(t) - r(t)\mathbf{1}$ is called the **risk premium** vector.

Definition

We say that a given, self-financing portfolio $\pi^*(\cdot)$ is an **arbitrage opportunity** if the corresponding wealth process satisfies $X^{\pi^*}(0) = 0$, $P(X^{\pi^*}(T) \geq 0) = 1$, and $P(X^{\pi^*}(T) > 0) > 0$. A financial market in which exists no arbitrage opportunity is called **viable**.

Theorem

If a market is viable, then there exists a measurable process $\theta(t)$, which satisfies

$$\sigma(t)\theta(t) = B(t), \text{ a.s.} \quad (1)$$

Conversely, if there exists process $\theta(t)$ satisfying (1), as well as

$$\int_0^T \|\theta(t)\|^2 dt < \infty \text{ a.s.}$$

$$E \left[\exp \left(- \int_0^T \theta(t)' dW(t) - \frac{1}{2} \int_0^T \|\theta(t)\|^2 dt \right) \right] = 1,$$

then the market is viable.

The process defined by (1) is called *the market price of risk*.

See [6], Theorem 4.2.

Definition

Let B be an \mathcal{F}_T measurable random variable, and let

$$X_0 := E \left[\frac{B}{S_0(T)} \right] < \infty. \quad (2)$$

- i) We say that B is **financiable** if there exists a portfolio $\pi^*(\cdot)$ such that $E[X^{\pi^*}(T)] = B$ a.s.
- ii) We say that market \mathcal{M} is **complete** if every \mathcal{F}_T measurable random variable B , satisfying (2), is financiable.

Theorem

A viable market is complete if and only if the number of stocks equals the dimension of the underlying Wiener process, and $\sigma(t)$ is nonsingular.

Merton portfolio problem

Definition

A function $U : (0, \infty) \rightarrow \mathbb{R}$, such that $U \in C^1$ is strictly concave and satisfies

$$U'(0) = \lim_{x \rightarrow 0^+} U'(x) = +\infty$$

$$U'(\infty) = \lim_{x \rightarrow \infty} U'(x) = 0,$$

is called a **utility function**.

Remark: Typical examples, for $x > 0$, of such utility functions are $U(x) = \frac{1}{\alpha}x^\alpha$ for $\alpha \in (0, 1)$, or $U(x) = \ln(x)$.

Definition

A nonnegative, \mathcal{F}_t measurable process $c(t)$ which satisfies

$$P\left(\int_0^T c(t) dt < \infty\right) = 1,$$

is called a **consumption process**. A pair $(\pi(t), c(t))$ consisting of the portfolio process and the consumption process is called an **admissible strategy** if

$$P(X^\pi(t) \geq 0) = 1, \quad \forall t \in [0, T].$$

The set of admissible strategies is denoted by \mathcal{A} .

Problem

The problem

$$\max_{(\pi, c) \in \mathcal{A}} E \left[\int_0^T U_1(c(t)) dt + U_2(X^\pi(T)) \right] \quad (3)$$

is called the (Merton's) continuous time portfolio problem.

Market Assumptions 4: Deterministic Coefficients

i) $r(t)$, $\sigma(t)$ and $\theta(t)$ are nonrandom, continuous functions of $t \in [0, T]$.

ii) $r(t)$ and $\|\theta(t)\|$ are Hölder continuous, i.e. there exist constants K and $\rho \in (0, 1)$ such that $|r(t) - r(u)| \leq K|t - u|^\rho$ and $|\|\theta(t)\| - \|\theta(u)\|| \leq K|t - u|^\rho$.

iii) There are constants c and d such that $0 < c \leq \|\theta(t)\| \leq d$ $\forall t \in [0, T]$.

iv) $I_1(\cdot)$ and $I_2(\cdot)$, defined by

$$I_1(y) := U_1'^{-1}(y)$$

$$I_2(y) := U_2'^{-1}(y),$$

have polynomial growth.

v) $U_1 \circ I_1(\cdot)$ and $U_2 \circ I_2(\cdot)$ have polynomial growth.

vi) $I_1(\cdot)$ is Hölder continuous.

vii) either $I_1'(\cdot)$ is defined and strictly negative, or $I_2'(\cdot)$ is defined and strictly negative.

Theorem

If the conditions i) to vii) are satisfied, then the portfolio that is the optimal solution of problem (3) is of the form

$$\pi^*(t) = \mathcal{K}(\cdot) (\sigma(t)\sigma(t)')^{-1} B(t), \quad \pi_0^*(t) = 1 - \pi^*(t)' \mathbf{1}$$

where $\mathcal{K}(\cdot)$ depends on the investor's preference structure given by $(U_1(\cdot), U_2(\cdot))$.

Thus, optimal investment strategies are the weighted averages of the bond and **Merton's portfolio** defined by

$$\pi_M(t) = (\sigma(t)\sigma(t)')^{-1} B(t).$$

Theorem

Two-fund Portfolio Separation Theorem *Suppose that one or both of the following two conditions is satisfied.*

- 1) *The expected utility of return is a hyperbolic absolute risk aversion (HARA) function.*
- 2) *The stocks' returns are multivariate normally distributed random variables.*







Then the optimal portfolio obtained by either the utility optimization, or the risk-reward approach, is a combination of the market portfolio and the riskless portfolio.

A **hyperbolic absolute risk aversion** (HARA) function is a function of the form:

$$U(x) = \frac{1 - \gamma}{\gamma} \left(\frac{ax}{1 - \gamma} + b \right)^\gamma,$$

where a , b , and γ satisfy appropriate conditions. See [5].

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