Modern Portfolio Theory: Part 2

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Measures of Risk Aversion

Absolute risk aversion

- An agent possesses risk aversion if and only if the utility function is concave.
- The higher the curvature of $U(x)$, the higher the risk aversion.
- A measure that stays constant with respect to linear transformations of a utility function is the **Arrow-Pratt measure of absolute risk-aversion (ARA)**, named after the economists Kenneth Arrow and John W. Pratt.

$$ARA\,(x) = -\frac{U''(x)}{U'(x)}.$$

In general, a utility function exhibits hyperbolic absolute risk aversion (HARA) if its absolute risk aversion is a hyperbolic function of wealth, i.e.

$$ARA\,(x) = -\frac{U''(x)}{U'(x)} = \frac{1}{ax+b}.$$

A solution to this differential equation is a function of the form:

$$U\,(x) = \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma}+b\right)^{\gamma},$$

where a, b, and γ satisfy the following conditions.

$$\gamma < 1, b > 0, \frac{ax}{1-\gamma}+b > 0.$$

See [10].

For the limiting case as $\gamma \to 1$, L'Hopital's Rule shows that the utility function becomes linear in wealth, and for the limiting case as $\gamma \to 0$, the utility function becomes logarithmic: $U(x) = \ln(x+b)$.

Relative risk aversion

The Arrow-Pratt measure of relative risk-aversion (RRA) is defined as

$$RRA(x) = -\frac{xU''(x)}{U'(x)}.$$

The notion of constant relative risk aversion has come under criticism from behavioral economics. According to Matthew Rabin of UC Berkeley, a consumer with a constant RRA who turns down gambles where he loses $100 or gains $110, each with 50% probability, will turn down 50-50 bets of losing $1,000 or gaining almost any sum of money.

EXAMPLES

- **Logarithmic utility** $U(x) = \ln x$ has $ARA(x) = \frac{1}{x}$ and $RRA(x) = 1.$
- **Power utility** $U(x) = \frac{1}{\gamma}x^{\gamma}$ $ARA(x) = \frac{1-\gamma}{x}$ and $RRA(x) = 1 - \gamma.$
- **Exponential utility** $U(x) = 1 - \exp(-ax)$ has $ARA(x) = a$ and $RRA(x) = ax.$
- Experimental and empirical evidence is mostly consistent with decreasing absolute risk aversion.
- If an investor experiences an increase in wealth, he/she will choose to increase (or keep unchanged, or decrease) the number of dollars invested in the risky asset held in the portfolio if absolute risk aversion is decreasing (or constant, or increasing).

Portfolio optimization in generalized Black-Scholes setting via general utility functions

Market Assumptions 5

- $Z(t) := \exp\left(-\int_0^t \theta(s)' dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right)$, $Z(0) = 1$, is a martingale.
- The state price density process $H_0(t) := \frac{Z(t)}{S_0(t)}$, $t \in [0, T]$.
- ξ is $\mathcal{F}_T$ measurable process that satisfies $E\left[\int_0^T H_0(t)\, c(t)\, dt + H_0(T)\, \xi\right] < X_0$.
- $M(t) := E\left[\int_0^T H_0(u)\, c(u)\, du + H_0(T)\, \xi \mid \mathcal{F}_t\right]$ has the martingale representation $M(t) = X_0 + \int_0^t \psi(u)\, dW(u)$.

Under the given assumptions, the optimal solution of Merton's portfolio optimization problem (see Problem 2, Part 1) is given by

$$\sigma(t)'\pi^*(t) = \frac{\psi(t)}{H_0(t)} + X(t)\,\theta(t).$$

If $U_1(x) = U_2(x) = \ln x$, then

$$\pi^*(t) = (\sigma(t)\sigma(t)')^{-1} B(t) X(t)\,.$$

See [11].

Application to mean-reverting assets.
F. E. Benth and K H. Karlsen. A Note on Merton's Portfolio Selection Problem for the Schwartz Mean-Reversion Model. *Stochastic Analysis and Applications*, Volume 23, Issue 4, 2005.

Market setting: $\frac{\mathrm{d}S(t)}{S(t)} = \beta(L - \ln S(t))\mathrm{d}t + \sigma \mathrm{d}W(t),\ S_0(t) = \mathrm{e}^{rt}$.

Approach: Define $V(t, X, S) = \sup_{\pi \in \mathbb{R}} E\left[U\left(X^{\pi}(T)\right) \mid \mathcal{F}_t\right]$.

Problem

$$\textit{Find } V(t, X, S), \quad V(T, X, S) = U(X), \quad V(T, 0, S) = 0.$$

Solution

V is found as the solution to an HJB equation, and the corresponding strategy is given by

$$\pi^*(t) = -\frac{\beta(L - \ln S(t) - r)\frac{\partial V}{\partial X} + \sigma^2 S(t)\frac{\partial^2 V}{\partial X \partial S}V}{\sigma^2 X \frac{\partial^2 V}{\partial X^2} V}.$$

Portfolio Optimization with Respect to Downside Risk Measures

Authors:

- G. Dmitrasinovic-Vidovic, see [2], [3], [4], [5], [6]
- A. Ware [3], [4], [5], [6]
- A. Lari-Lavassani [4], [5]
- X. Li [4], [5]
- S. Emmer, C. Klüppelberg, and R. Korn [8]

Downside Risk Measures

Definition

Let $\mathcal{V}$ be the set of real valued random processes on a complete probability space $(\Omega, \mathcal{F}, P)$, and let $V(t) \in \mathcal{V}$. We define[a]

α-quantile $q_v := \inf\{u \in R | P(V(t) \leq u) \geq \alpha\}$.

Capital at risk CaR$(V(t)) := V(0)R_0(t) - q_v$.

Value at risk VaR$(V(t)) := E\left[V(t)\right] - q_v$

Tail mean TM$_\alpha(V(t)) := E[V(t) \mid V(t) \leq q_v]$.

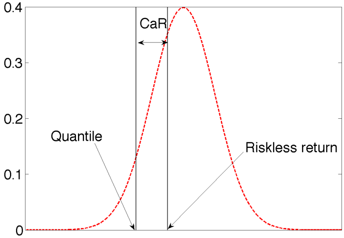
Conditional capital at risk

$$\text{CCaR}(V(t)) := V(0)R_0(t) - TM_\alpha(V(t)).$$

Conditional value at risk CVaR$(V(t)) := E\left[V(t)\right] - TM_\alpha(V(t))$.

[a] $R_0(t) := \exp\left(\int_0^t r(s)ds\right)$

Comparison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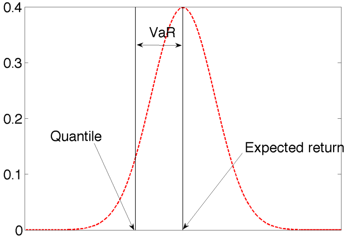
0
CaR
Quantile
Riskless return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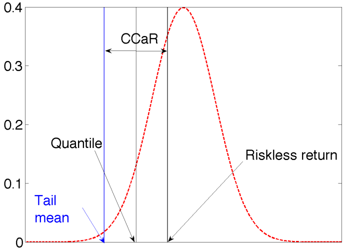
VaR
Quantile
Expected return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CCaR
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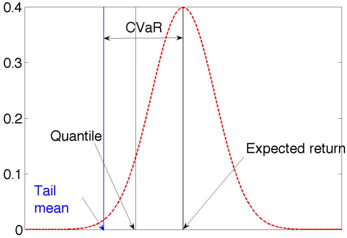
Riskless return
Tail
mean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CVaR
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Expected return
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mean

Mean-Quantile Efficient Portfolios of Log-normally Distributed Assets

- Assumptions: 1 to 3, 5, 6 from Market Assumptions 3, Part 1, hold.
- Assumption 4. $\sigma(t)$, $\sigma^{-1}(t)$, $b(t)$ and $r(t)$ are deterministic, Lebesgue measurable, bounded functions over $[0, T]$.
- Wealth $X^{\pi}(t) = \sum_{i=0}^{m} N_i(t) S_i(t)$.
- Portfolio $\pi_i(t) = \frac{N_i(t) S_i(t)}{X^{\pi}(t)}, \quad i = 1, \cdots, m.$
- $\mathcal{Q}$ is the set of admissible portfolios, i.e., portfolios that are deterministic and bounded.

Problem 1

$$\min_{\pi(\cdot)\in\mathcal{Q}} \rho(X_0, \pi, t),$$

where $\rho : L^2([0,t];\mathbb{R}^m) \to \mathbb{R}$ is one of downside risk measures.

Problem 2

$$\max_{\pi(\cdot)\in\mathcal{Q}} E\left[X(T)\right], \quad \text{subject to} \quad \rho(X_0, \pi, T) \leq C.$$

Approach: We project the optimization problems considered in this paper onto the family of surfaces

$$\mathcal{Q}_\varepsilon = \left\{ \pi(\cdot) \in L^2([0,T];\mathbb{R}^m) : \|\sigma'\pi\|_T^2 = \varepsilon^2 \right\} \quad \text{and} \quad \mathcal{Q} = \bigcup_{\varepsilon \geq 0} \mathcal{Q}_\varepsilon .$$

This leads to

$$\max_{\varepsilon \geq 0} \min_{\pi \in \mathcal{Q}_\varepsilon} \rho(X_0, \pi, T)$$

Solution

$$\pi^*(t) = \varepsilon_\rho (\sigma(t)\sigma(t)')^{-1} B(t),$$

where ε_ρ depends on the the choice of risk measure and the acceptable risk level. (Credit to [8].)

Mean-Quantile Efficient Portfolios of Mean-reverting Assets

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \beta_i(L_i - c_i \ln S_i(t))\mathrm{d}t + \sum_j \sigma_{ij}\mathrm{d}W_j(t), S_i(0) > 0, i = 1, \dots, m.$$

- the mean-reversion rate $\beta_i > 0$,
- $c_i = 0$ *or* 1, $i = 1, \cdots, m$, are scaling factors,
- $L_i \in \mathbb{R}$, long term mean *of* $\ln S_i(t)$.
- Notation

$$a_i = \beta_i L_i, \;\; b_i = \beta_i c_i, \;\; i = 1, \dots, m, \quad \boldsymbol{\sigma_i} = (\sigma_{i1}, \dots, \sigma_{im})\,.$$

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = (a_i - b_i \ln S_i(t))\mathrm{d}t + \boldsymbol{\sigma_i} d\boldsymbol{W}(t), \quad i = 1, \dots, m.$$

We make the simplifying assumption that our portfolio π is constant in time.
Define

$$\hat{a}_i = a_i - \frac{1}{2} \left\| \boldsymbol{\sigma_i} \right\|^2 ,$$

$$\mathcal{E}\left(t,b\right) := \int_0^t \mathrm{e}^{-sb}\mathrm{d}s = \begin{cases} \frac{1-\mathrm{e}^{-tb}}{b} & \text{if } b \neq 0 \\ t & \text{if } b = 0, \end{cases}$$

$$\mathcal{A}_i = \mathcal{E}\left(t,b_i\right)\left(\hat{a}_i - b_i Y_i(0)\right) - \hat{a}_i t, \;\; i = 1, \cdots m.$$

Variance-covariance matrix

$$\mathbf{F}(t,\boldsymbol{b}) = \boldsymbol{\sigma}\boldsymbol{\sigma}' \,.\, \mathcal{E}(t,\boldsymbol{b}).$$

Risk premium vector

$$\boldsymbol{g}(t) := (\boldsymbol{a} - r\mathbf{1})t + \boldsymbol{\mathcal{A}}(t,\boldsymbol{b}).$$

$$f(\boldsymbol{\pi},t) : = \boldsymbol{\pi}'\boldsymbol{g}(t) - \frac{t}{2} \left\| \boldsymbol{\pi}'\boldsymbol{\sigma} \right\|^2 - |z_\alpha| \sqrt{\boldsymbol{\pi}'\mathbf{F}(t,\boldsymbol{b})\boldsymbol{\pi}}.$$

Expected wealth

$$\mathbb{E}[X^{\pi}(t)] = X_0 \exp\left(rt + \boldsymbol{\pi}'\boldsymbol{g}(t) - \frac{1}{2}\boldsymbol{\pi}'\widetilde{\boldsymbol{F}}(t,\boldsymbol{b})\boldsymbol{\pi}\right),$$

where

$$\widetilde{\boldsymbol{F}}(t,\boldsymbol{b}) = \boldsymbol{F}(t,\boldsymbol{0}) - \boldsymbol{F}(t,\boldsymbol{b}).$$

Market price of risk

$$\theta\,(t) = \mathbf{F}(t,\boldsymbol{b})^{-1}\boldsymbol{g}(t).$$

Capital at risk

$$\mathrm{CaR}(\boldsymbol{\pi},t) = X_0 \mathrm{e}^{rt}\left(1 - \mathrm{e}^{f(\boldsymbol{\pi},t)}\right).$$

Problem

$$\min_{\boldsymbol{\pi}\in\mathbb{R}^m} CaR(\boldsymbol{\pi}, T) \tag{1}$$

Theorem

1) If $\boldsymbol{g}(T)'\mathbf{F}^{-1}\boldsymbol{g}(T) > \|z_\alpha\|^2$*, the optimal solution of problem (1) is equal to*

$$\boldsymbol{\pi}^* = \left(T\boldsymbol{\sigma}\boldsymbol{\sigma}' + \frac{\|z_\alpha\|}{\lambda^*}\mathbf{F}\right)^{-1}\boldsymbol{g}(T),$$

where λ^* *is the unique positive solution of the equation*

$$\left\|\mathbf{A}\left(\lambda T\boldsymbol{\sigma}\boldsymbol{\sigma}' + \|z_\alpha\|\,\mathbf{F}\right)^{-1}\boldsymbol{g}(T)\right\| = 1,$$

and $\mathbf{A}$ *is a unique square root of* $\mathbf{F}$*.*
2) If $\boldsymbol{g}(T)'\mathbf{F}^{-1}\boldsymbol{g}(T) \leq |z_\alpha|^2$ *the optimal solution of problem (1) is* $\boldsymbol{\pi} = \mathbf{0}$*.*

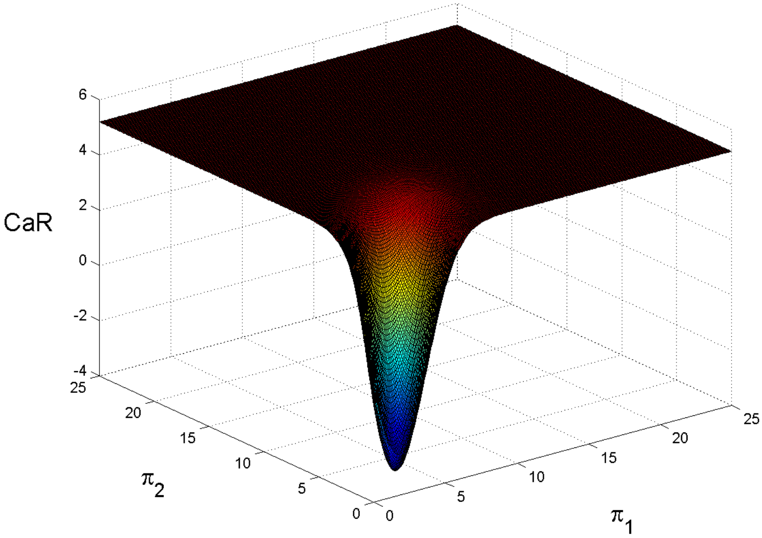
Example

$$T = 2,\ X_0 = 5,\ r = 0.02,\ z_\alpha = 1.645,\ \boldsymbol{\beta} = \boldsymbol{b} = (2,1)',$$
$$\boldsymbol{L} = (1.2, 2.6)',\ \ \ln \boldsymbol{S}(0) = (1.5, 2),\ \boldsymbol{\sigma}\boldsymbol{\sigma}' = \begin{pmatrix} 1 & -0.6 \\ -0.6 & 1 \end{pmatrix}.$$

Results
Risk premium $\boldsymbol{g}(T) = (-0.21, 1.02)'$,
Market price of risk $\theta\,(T) = 3.001$

Minimum capital at risk: $\mathrm{CaR}(\boldsymbol{\pi}^*, T) = -3.3834$
Optimal portfolio: $\boldsymbol{\pi}^* = (1.2, 2.4)'$,

Maximum unconstrained wealth $\mathbb{E}[X^\pi(t)] = 28.1809$
Optimal portfolio: $\boldsymbol{\pi}^* = (1.9, 3.7)'$,


CaR
6
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2
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-2
-4
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15
10
5
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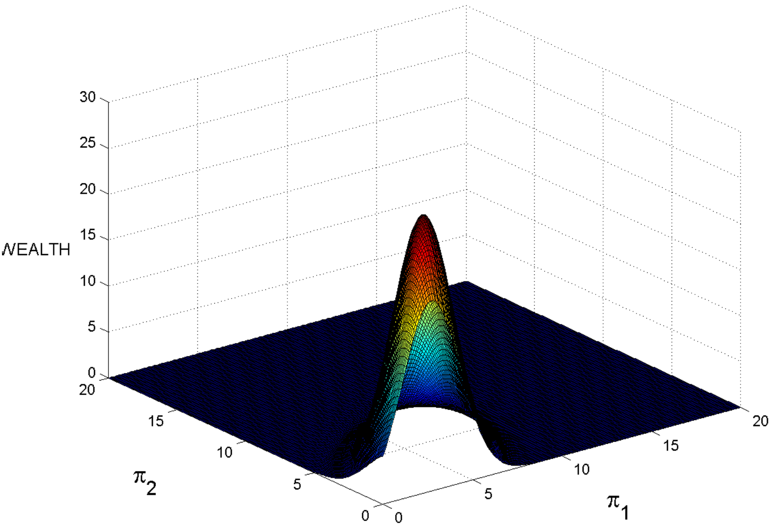
π2
0
5
10
15
20
25
π1


WEALTH
30
25
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15
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5
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20
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10
5
0
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5
10
15
20
π_2
π_1

Further research

- Asymptotic behaviour of mean reverting optimal portfolios

T	1	2	3	4	5	6	7	8	9
π_1	1.6	1.9	2	2	2	2	2	2	2
π_2	5.2	3.7	3.2	2.9	2.8	2.7	2.6	2.5	2.5
$\mathbb{E}[X]$	20	28	40	58	84	123	182	270	403
CaR	4.6	-2.3	-22	-67	-180	-436	-977	-2043	-5047

- Calibration

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