Generalization of Black-76 formula: Markov-modulated Volatility

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Outline of Presentation

- Introduction
- Literature Review
- Generalization of Black-76 Model
- Conclusion
- Future Problems

Introduction

Goal of this paper:

- Get generalization of Black-76 model
- To price European call option written on electricity forward price

- Markov-modulated models have been extensively studied in the literature.
- The term regime-switching is used to describe such models.
- A potentially useful approach to model nonlinearities in time series is to assume different behavior (structural break) in one subsample(or regime) to another.

Introduction

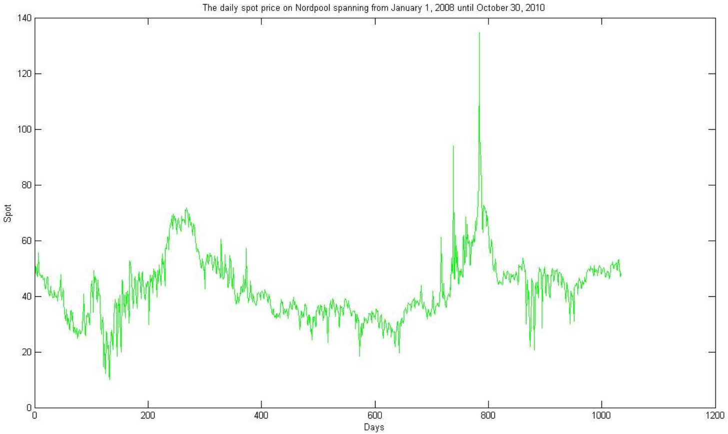
- Elliott, R.J. and Buffington, T.R., *American options with regime switching*, 2002
- Elliott, R.J. and Valchev, S., *Libor Market Model with Regime-Switching Volatility*, 2004
- Elliott, R.J. and Swishchuk, A. V., *Pricing Options and Variance Swaps in Markov-Modulated Brownoan Markets*, 2004
- Weron, R., *Heavy-tails and regime-switching in electricity prices*, 2008
- Benth, F.E., et al, *HMM filtering and parameter estimation of an electricity spot price model*, 2010

Introduction

- Electricity is a very unique commodity:
  - price spkies/jumps
  - cannot be stored
  - strong weather and business cycle dependence
  - power plant outages or tranmission grid unreliability add complexity and randomness

Daily SpotPrice

Data from Nordpool Jan 1 2008 - Oct 30 2010



Black-76 Model

- It was first presented in a paper written by Fischer Black in 1976.
- It is a variant of the Black-Scholes option pricing model.
- Non-randomness is a challenge in option pricing for commodity prices.
- While it is not reasonable to model the spot price with a Brownian motion, it may be reasonable to model the forward price with one.

Black-76 Formula

The price of a call option at time $t \leq T$, written on a forward with delivery at time τ, where the option has exercise time $T \leq \tau$ and strike price K, is

$$C(t;T,K,\tau) = e^{-r(T-t)}\{f(t,\tau)\Phi(d_1) - K\Phi(d_2)\}. \qquad (1)$$

Here,

$$d_1 = \frac{ln(f(t,\tau)/K) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{ln(f(t,\tau)/K) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$

and Φ is the cumulative standard normal probability distribution function.

Markov-Modulated Volatility: $\sigma \equiv \sigma(Z(t))$

- $\sigma \equiv \sigma(Z(t))$ is the stochastic volatility driven by Markov process
- The implied volatility is not constant fails the familiar log-Brownian paradigm in various ways
- The simplest way to introduce additional randomness is to let the volatility be function of a finitestate Markov chain
- In applications, it is likely that the number of states of the Markov chain will be small

Pricing options for Markov-modulated markets without jumps

In here, we are going to use the risk-neutral dynamics for forward contract with Markov-Modulated Volatility

$$\frac{df(t,\tau)}{f(t,\tau)} = \sigma(Z(t))dW(t), \tag{2}$$

where $W(t)$ is the Brownian motion under the risk-neutral probability Q, to get generalization of the famous Black-76 Formula

Theorem 2

The price of a call option at time $t \leq T$, written on a forward with delivery at time τ, where the option has exercise time $T \leq \tau$ and strike price K is

$$C_T^Z(\tau) = \int_0^\infty C_T^B\left(\left(\frac{z}{T}\right)^{1/2}, T\right) F_T^Z(dz) \tag{3}$$

- $F_T^Z(dz)$ is the distribution of the random variable $Z_T \equiv \int_0^T \sigma^2(Z_s)ds$
- $C_T^B(\sigma, T)$ is the Black-76 formula for call option written on forward at delivery τ, as defined in (1).

Collary 1

For the case where $E = \{1, 2\}$ the distribution F_T of the random variable Z_T can be expressed in an explicit form. Indeed, let $\nu(t)$ be a counting process of jumps of Y. Then

$$Z_T = \int_0^T [\sigma^2(1)I(Z_t = 1) + \sigma^2(2)I(Z_t = 2)]dt = a \cdot T + b \cdot J_T$$

where

$$J_T = \int_0^T (-1)^{\nu(t)} dt$$

$$a = \frac{1}{2}(\sigma^2(1) + \sigma^2(2)),$$

$$b = \frac{1}{2}(\sigma^2(1) - \sigma^2(2)).$$

Collary 1

We consider the symmetric case where transition intensities are equal: $q_{12} = q_{21} = \lambda$. The distribution G_T of the random variable J_T is given by

$$G_T(dz) = e^{-\lambda T}\epsilon_T(dz) + h_{T,\lambda}(z)dz,$$

$$h_{T,\lambda}(z) = \lambda/2e^{-\lambda T}\Big[I_0(\lambda(T^2 - z^2)^{\frac{1}{2}}) + \left(\frac{T+z}{T-z}\right)^{\frac{1}{2}} I_1(\lambda(T^2 - z^2)^{\frac{1}{2}})\Big]\mathbb{I}_{(-T,T)}(z),$$

where ϵ_T is the unit mass at T and the modified Bessel functions of the first kind I_0 and I_1 are defined by

$$I_0(s) = \sum_{k=0}^{\infty} \frac{(s^2/4)^k}{k!^2}, \qquad I_1(s) = \frac{s}{2}\sum_{k=0}^{\infty} \frac{(s^2/4)^k}{k!(k+1)!}$$

Collary 2

For the case where $E = \{1, 2\}$ the distribution F_T of the random variable Z_T can be expressed in an explicit form. Indeed, let $\nu(t)$ be a counting process of jumps of Y. Then

$$Z_T = \int_0^T [\sigma^2(1)I(Z_t = 1) + \sigma^2(2)I(Z_t = 2)]dt = a \cdot T + b \cdot J_T$$

where

$$J_T = \int_0^T (-1)^{\nu(t)} dt$$

$$a = \frac{1}{2}(\sigma^2(1) + \sigma^2(2)),$$

$$b = \frac{1}{2}(\sigma^2(1) - \sigma^2(2)).$$

Collary 2

We consider the non-symmetric case where transition intensities are: $q_{12} = \lambda$ and $q_{21} = \kappa$. The distribution G_T of the random variable J_T is given by

$$G_T(dz) = e^{-\lambda T}\epsilon_T(dz) + h_{T,\lambda,\kappa}(z)dz,$$

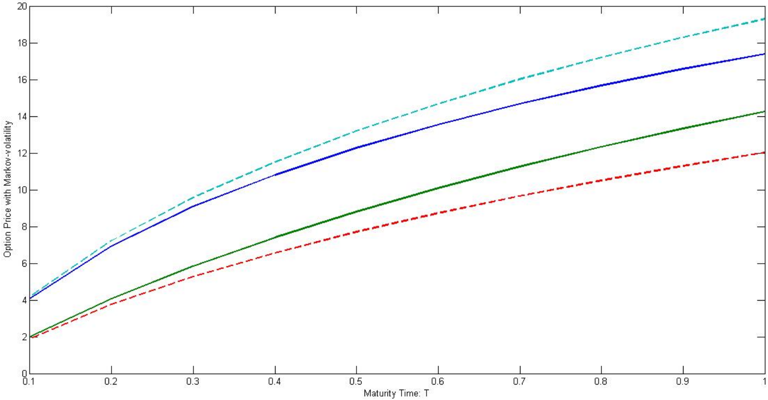
$$h_{T,\lambda,\kappa}(z) = \lambda/2e^{-(\lambda+\kappa)T/2-(\lambda-\kappa)z/2}\Big[I_0(\lambda\kappa(T^2 - z^2)^{\frac{1}{2}}) + (\frac{\kappa}{\lambda})^{\frac{1}{2}}\left(\frac{T+z}{T-z}\right)^{\frac{1}{2}} I_1(\lambda\kappa(T^2 - z^2)^{\frac{1}{2}})\Big]\mathbb{I}_{(-T,T)}(z),$$

where ϵ_T is the unit mass at T and the modified Bessel functions of the first kind I_0 and I_1 are defined by

$$I_0(s) = \sum_{k=0}^{\infty} \frac{(s^2/4)^k}{k!^2}, \qquad I_1(s) = \frac{s}{2}\sum_{k=0}^{\infty}\frac{(s^2/4)^k}{k!(k+1)!}$$

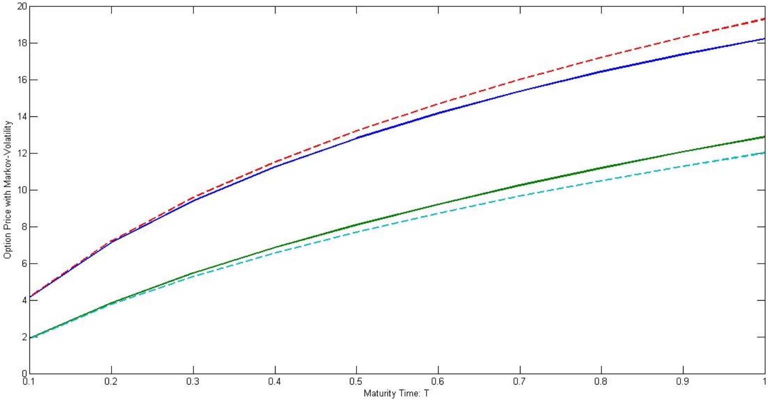
Synthetic Data Without Jump: Symmetric

Option Price with Markov Volatility. The parameter values: $f_0 = 50$, $K = 60$, $\lambda = 0.8$, $r = 0.04$, $\sigma_1 = 1.2$, $\sigma_2 = 0.8$



Synthetic Data Without Jump: NonSymmetric

Option Price with Markov Volatility. The parameter values: $f_0 = 50$, $K = 60$, $\lambda = 0.3$, $\kappa = 0.1$, $r = 0.04$, $\sigma_1 = 1.2$, $\sigma_2 = 0.8$



- Y_1, Y_2, ..., $Y_{\nu(t)}$ are i.i.d. R.V. with values in $(-1, +\infty)$
- $\nu(t)$ is a Poisson process with intensity $\lambda > 0$
- τ_1, τ_2, ..., $\tau_{\nu(t)}$ are random moments of time
- $\nu(t)$, $(Y_i; i \geq 1)$ and $(\tau_i; i \geq 1)$ are independent on $Z(t)$ and $W(t)$

Here, we are going to use the risk-neutral dynamics for forward contract with Markov-Modulated volatility with compound geometric Poisson process.
Such dynamics is described by the following stochastic equation:

$$\frac{df(t,\tau)}{f(t,\tau)} = \sigma(Z(t))dW(t) + \int_{-1}^{+\infty} y\nu(dt,dy), \tag{4}$$

where $W(t)$ is the Brownian motion under the risk-neutral probability Q.

Solution of this equation may be given in the form

$$f(t,\tau) = f(0,\tau)\exp\left\{(\lambda\int_{-1}^{+\infty}\ln(1+y)H(dy)ds + \sigma(Z(s)) - \frac{1}{2}\int_0^t \sigma^2(Z(s)))dW(s) + \int_{-1}^{+\infty}\ln(1+y)\tilde{\nu}(ds,dy)\right\}, \tag{5}$$

$$f^*(t,\tau) = f(0,\tau)\exp\left\{(-\int_0^t r(Z(s)) - \frac{1}{2}\sigma^2(Z(s)))ds + \int_0^t \sigma(Z(s))dW^*(s)\right\}\prod_{k=1}^{N(t)}(1+Y_k). \tag{6}$$

where

$$W^*(t) := W(t) - \int_0^t \frac{r(Z(s))}{\sigma(Z(s))}ds \tag{7}$$

Formula for the price of contingent claim $g_T(f(t,\tau))$

Theorem 3

The price $C^g_{T,Z,s}$ of contingent claim $g_T(f(t,\tau))$ in zero moment of time with expiry date T has the form:

$$\begin{aligned} C^g_{T,Z,s} &= \mathbb{E}^*_{T,Z,s}\Big[g_T(f(t,\tau))\exp\Big\{-\int_0^T r(Z(s))ds\Big\}\Big] \\ &= \mathbb{E}^*_{T,Z,s}\Big[g_T\Big(f^*(t,\tau)\exp\Big\{\int_0^T r(Z(s))ds\Big\}\Big)\exp\Big\{-\int_0^T r(Z(s))ds\Big\} \end{aligned} \tag{8}$$

Formula for the price of European Call

In the case $g_T(f(t,\tau)) = (f(t,\tau) - K)^+$, where K is a strike price.

Theorem 4

$$\begin{aligned} C_{T,Z,s} &= \mathbb{E}^*_{T,Z,s}\Big[(f(t,\tau) - K)^+ \exp\Big\{-\int_0^T r(Z(s))ds\Big\}\Big] \\ &= \mathbb{E}^*_{T,Z,s}\Big[(f^*(t,\tau)\exp\Big\{-\int_0^T r(Z(s))ds\Big\} - K)^+ \\ &\times \exp\Big\{-\int_0^T r(Z(s))ds\Big\}\Big], \end{aligned} \tag{9}$$

where process $f^*(t,\tau)$ is defined in (6).

Corollary 3

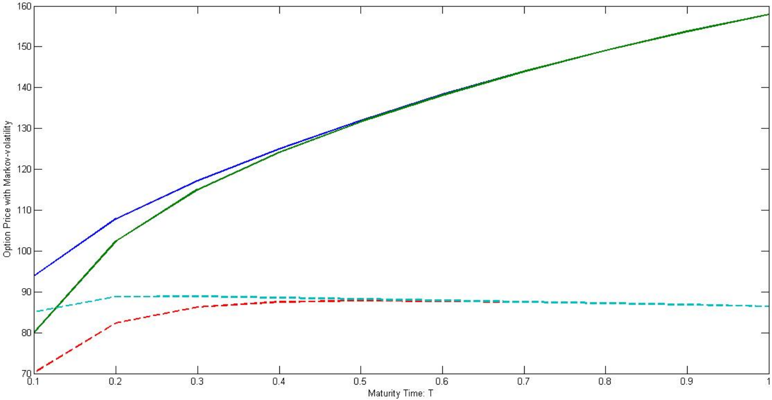
Let $g_T(f(t,\tau)) = (f(t,\tau) - K)^+$, and let $r \equiv 0$. Then from Theorem 4, formula (25), and Black-76 value C_T^{BS} for European call option it follows that the price $C_{T,Z,s}$ of contingent claim has the form

$$C_{T,Z,s} = \sum_{k=0}^{+\infty} \frac{\exp -\lambda T(\lambda T)^k}{k!} \times \int_{-1}^{+\infty} \cdots \int_{-1}^{+\infty} C_T^{BS}((\frac{z}{T})^{\frac{1}{2}}, T, f \prod_{i=1}^{k}(1+y_i))F_T(dz)) \times H^*(dy_1) \times \ldots H^*(dy_k) \tag{10}$$

where function C_T^{BS} is a Black-76 value for European call option.

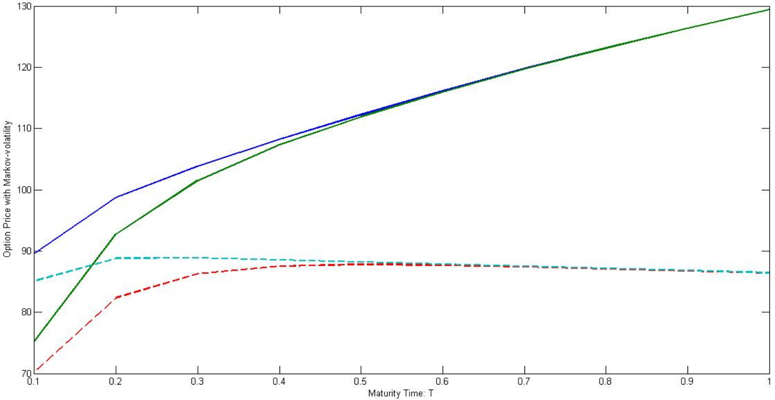
Synthetic Data With Jumps: Symmetric

Option Price with Markov Volatility. The parameter values: $f_0 = 90$, $K = 100$, $\lambda = 0.8$, $r = 0.04$, $\sigma_1 = 12.5$, $\sigma_2 = 8$



Synthetic Data With Jumps: Non-Symmetric

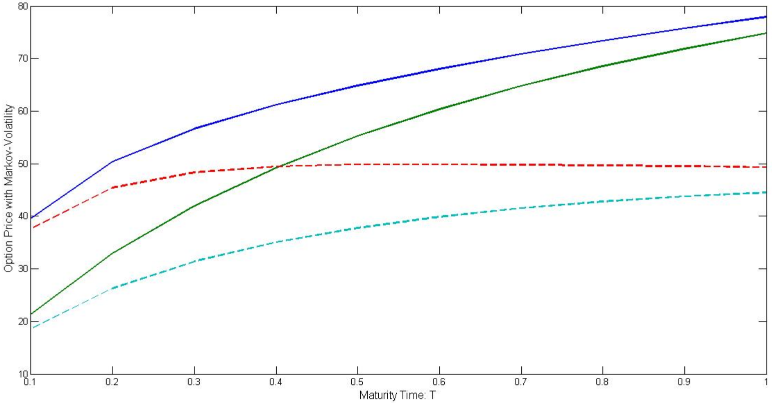
Option Price with Markov Volatility. The parameter values: $f_0 = 90$, $K = 100$, $\lambda = 0.4$, $\kappa = 0.1$, $r = 0.04$, $\sigma_1 = 12.5$, $\sigma_2 = 8$



- One of the most famous markets to trade electricity
- Daily electricity spot prices from Nordpool
- Weekly elctricity forward prices
- Two states markov chain
- we extract the jumps from the original series of electricity spot prices by writing a numerical algorithm

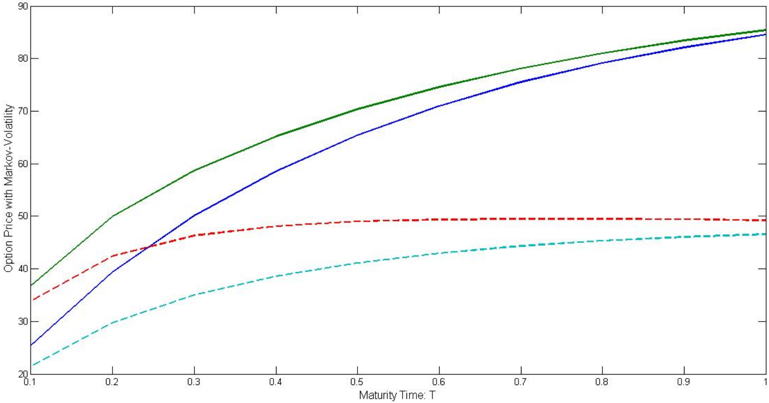
Applications: Data from Nordpool

Option Price with Markov Volatility. The parameter values: $f_0 = 51.3375$, $K = 60$, $\lambda = 0.5$, $r = 0.04$, $\sigma_1 = 7.3683$, $\sigma_2 = 3.3916$



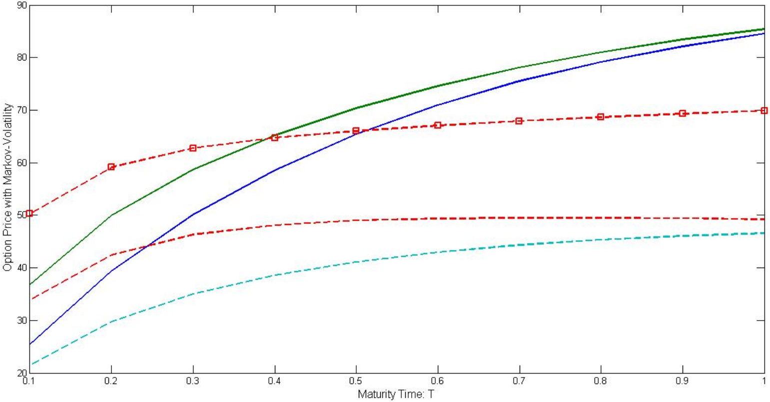
Applications: Data from Nordpool

Option Price with Markov Volatility. The parameter values: $f_0 = 51.3375$, $K = 60$, $\lambda = 0.823$, $\kappa = 0.177$, $r = 0.04$, $\sigma_1 = 7.3683$, $\sigma_2 = 3.3916$



Comparision

Comparison between Markov Driven Volatility and Black-76 + Jumps



Conclusion

- A closed form formula for generlize Black-76 with Markov-modulated volatility option pricing model has been developed
- In the case of two states markov chain, we have an exact analytic formula for both symmetric and nonsymmetric case.
- By adding an extra randomness into the volatility, we are able to capture the important characteristic of electricity prices, the spike.
- An exact closed formula provides useful insight for the European option pricing in the Black model with Markov-modulated volatility, it is not only explains the effect of markov model, but accelerates the computation of the European option pricing in the Black model with Markov-modulated volatility.

Future Problems

- Consider more states for Markov Chain.
- Produce results for forward contracts with different maturity times.
- Apply our model to Alberta energy market.

Questions?

Thank you for you attention.