

Paper Review
Analysis of Valuation Formulae and Applications
to Exotic Options in Lévy Models
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Lévy process

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, where $\mathcal{F} = \mathcal{F}_T$ and the filtration $\mathbb{F} = (\mathcal{F})_{t \in [0, T]}$ satisfies the usual conditions

Definition

A càdlàg, adapted, real valued stochastic process $L = (L_t)_{t \geq 0}$ with $L_0 = 0$ a.s. is called a *Lévy process* if the following conditions are satisfied:

- (L1): L has *independent increments*, i.e. $L_t - L_s$ is independent of \mathcal{F}_s for any $0 \leq s < t \leq T$.
- (L2): L has *stationary increments*, i.e. for any $s, t \geq 0$ the distribution of $L_{t+s} - L_t$ does not depend on t .
- (L3): L is *stochastically continuous*, i.e for every $t \geq 0$ and $\epsilon > 0$:

$$\lim_{s \rightarrow t} P(|L_t - L_s| > \epsilon) = 0.$$

Exponential Semimartingale Model

- Consider models of the form

$$S_t = S_0 e^{H_t} \quad 0 \leq t \leq T,$$

where $H = (H_t)_{0 \leq t \leq T}$ is a semimartingale with $H_0 = 0$.

- Assume that P is an (equivalent) martingale measure for the asset S
- Assume zero interest rates
- The *underlying process* can be the log-asset price process or the supremum of the log-asset price process or an average of the log-asset price process. This process will always be denoted by X .
- The *payoff function* is an arbitrary function $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$

Valuation Formula

(C1): Assume that $g \in L^1_{bc}(\mathbb{R})$.

(C2): Assume that $M_{X_T}(R)$ exists.

(C3): Assume that $\hat{g} \in L^1(\mathbb{R})$.

Theorem

If the asset price process is modeled as an exponential semimartingale process and conditions (C1) – (C3) are in force, then the time-0 price function $V_f(X; s)$ of an option on $S = (S_t)_{0 \leq t \leq T}$ with payoff f is given by

$$V_f(X; s) = \frac{e^{-Rs}}{2\pi} \int_{\mathbb{R}} e^{ius} \varphi_{X_T}(-u - iR) \hat{f}(u + iR) du. \quad (2.12)$$

Valuation Formula - Assumption C3

Let us consider the Sobolev space $H^1(\mathbb{R})$, with

$$H^1(\mathbb{R}) = \{g \in L^2(\mathbb{R}) \mid \partial g \text{ exists and } \partial g \in L^2(\mathbb{R})\},$$

where ∂g denotes the *weak* derivative of a function g .

Lemma

Let $g \in H^1(\mathbb{R})$, then $\hat{g} \in L^1(\mathbb{R})$.

Valuation Formula

Define the function \bar{g} and the measure ϱ :

$$\bar{g}(x) := g(-x) \text{ and } \varrho(dx) := e^{Rx} P_{X_T}(dx). \quad (2.19)$$

(D1): Assume that $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

(D2): Assume that $M_{X_T}(R)$ exists ($\Leftrightarrow |\varrho| < \infty$).

Theorem

Let the asset price process be modeled as an exponential semimartingale process and conditions (D1) – (D2) be in force.

The time-0 price function $V_f(X; s)$ of an option on $S = (S_t)_{0 \leq t \leq T}$ with payoff f is given by

$$V_f(X; s) = \lim_{A \rightarrow \infty} \frac{e^{-Rs}}{2\pi} \int_{-A}^A e^{-ius} \varphi_{X_T}(u - iR) \hat{f}(iR - u) du, \quad (2.20)$$

at the point $s \in \mathbb{R}$, if $V_f(X; \cdot)$ is of bounded variation in a neighbourhood of s , and $V_f(X; \cdot)$ is continuous at s .

Options on Several Assets

Typical examples of options on several assets are **basket options** and **options on the minimum or maximum of several assets**, with payoff

$$(S_T^1 \wedge \cdots \wedge S_T^d - K)^+;$$

typical examples of options on functionals of a single asset are **barrier options**, with payoff

$$(S_T - K)^+ 1_{\{\bar{S}_T > B\}},$$

and **slide-in** or **corridor** options, with payoff

$$(S_T - K)^+ \sum_{i=1}^N 1_{\{L < S_{T_i} < H\}},$$

at maturity T , where $0 = T_0 < T_1 < \cdots < T_N = T$.

Options on Several Assets (continued)

(A1): Assume that $g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

(A2): Assume that $M_{X_T}(R)$ exists.

(A3): Assume that $\hat{g} \in L^1(\mathbb{R}^d)$.

Theorem

If the asset price processes are modeled as exponential semimartingale processes and conditions (A1) – (A3) are in force, then the time-0 price function $\mathbb{V}_f(X; s)$ of an option on S with payoff f is given by

$$\mathbb{V}_f(X; s) = \frac{e^{-\langle R, s \rangle}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u, s \rangle} M_{X_T}(R + iu) \hat{f}(iR - u) du. \quad (3.4)$$

Options on Several Assets (continued)

(G1): Assume that $g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

(G2): Assume that $M_{X_T}(R)$ exists.

Theorem

If the asset price process is modelled as an exponential semimartingale process and conditions (G1) – (G2) are in force, then the time-0 price function $\mathbb{V}_f(X; s)$ of an option on $S = (S_t)_{0 \leq t \leq T}$ with payoff f satisfies

$$\mathbb{V}_f(X; \cdot) = \frac{e^{-\langle R, \cdot \rangle}}{(2\pi)^d} L^2\text{-}\lim_{A \rightarrow \infty} \int_{[-A, A]^d} e^{-i\langle u, \cdot \rangle} \varphi_{X_T}(u - iR) \hat{f}(iR - u) du.$$

Sensitivities

Option price as a function of the initial value:

$$V_f(X; S_0) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-iu} M_{X_T}(R-iu) \hat{f}(u+iR) du$$

Delta:

$$\Delta_f(X; S_0) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-1-iu} M_{X_T}(R-iu) \frac{\hat{f}(u+iR)}{(R-iu)^{-1}} du.$$

Gamma:

$$\Gamma_f(X; S_0) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-2-iu} \frac{M_{X_T}(R-iu) \hat{f}(u+iR)}{(R-1-iu)^{-1} (R-iu)^{-1}} du.$$

Infinum/Supremum Processes

Assumption (EM).

There exists a constant $M > 1$ such that

$$\int_{|x|>1} e^{ux} \lambda(dx) < \infty, \quad \forall u \in [-M, M].$$

Using (EM) and Theorems 25.3 and 25.17 in Sato (1999), we get that

$$E[e^{uL_t}] < \infty, \quad E[e^{u\bar{L}_t}] < \infty \text{ and } E[e^{u\bar{L}_t}] < \infty$$

for all $u \in [-M, M]$.

Infinum/Supremum Processes (continued)

Corollary

Let L be a Lévy process that satisfies assumption (EM). Then, the characteristic function $\varphi_{\bar{L}_t}$ of \bar{L}_t , resp. $\varphi_{\underline{L}_t}$ of \underline{L}_t , possesses a continuous extension

$$\varphi_{\bar{L}_t}(z) = \int_{\mathbb{R}} e^{izx} P_{\bar{L}_t}(dx), \text{ resp. } \varphi_{\underline{L}_t}(z) = \int_{\mathbb{R}} e^{izx} P_{\underline{L}_t}(dx),$$

to the half-plane $z \in \{z \in \mathbb{C} : -M \leq \Im z\}$, resp. $z \in \{z \in \mathbb{C} : \Im z \leq M\}$, that is analytic in the interior of the half-plane $\{z \in \mathbb{C} : -M < \Im z\}$, resp. $\{z \in \mathbb{C} : \Im z < M\}$.

Infinum/Supremum Processes (continued)

Lemma

Let $L = (L_t)_{0 \leq t \leq T}$ be a Lévy process that satisfies assumption (EM) and $\theta \sim \text{Exp}(q)$ be independent of the process L . If $q > \bar{\alpha}(M) \vee \underline{\alpha}(M)$, then the characteristic function $\varphi_{\bar{L}_\theta}$ of \bar{L}_θ possesses a continuous extension

$$\varphi_{\bar{L}_\theta}(z) = \int_{\mathbb{R}} e^{izx} P_{\bar{L}_\theta}(dx) = q \int_0^\infty e^{-qt} E[e^{iz\bar{L}_t}] dt \quad (5.6)$$

to the half-plane $z \in \{z \in \mathbb{C} : -M \leq \Im z\}$, that is analytic in the interior of the half-plane, $\{z \in \mathbb{C} : -M < \Im z\}$.

Infinum/Supremum Processes (continued)

Let L be a Lévy process and θ an independent, exponentially distributed time with parameter q ; then we have that

$$E[e^{izL_\theta}] = E[e^{iz\bar{L}_\theta}]E[e^{izL_\theta}]$$

or equivalently,

$$\frac{q}{q - \psi(z)} = \varphi_q^+(z)\varphi_q^-(z), \quad z \in \mathbb{R};$$

here ψ denotes the characteristic exponent of L and φ_q^+, φ_q^- are the so-called Wiener-Hopf factors.

Infinum/Supremum Processes (continued)

Let us introduce the following notation:

$$\bar{\alpha}(M) := M|b| + \frac{1}{2}cM^2 + \int_{\mathbb{R}} |e^{Mx} - 1 - Mx| \lambda(dx)$$

$$\underline{\alpha}(M) := M|b| + \frac{1}{2}cM^2 + \int_{\mathbb{R}} |e^{-Mx} - 1 + Mx| (dx)$$

$$\alpha^*(M) := \max\{\bar{\alpha}(M), \underline{\alpha}(M), \phi(M)\},$$

where ϕ denotes the cumulant associated to L , i.e.

$$\phi(u) = \log E[e^{uL_1}].$$

Infinum/Supremum Processes (continued)

Theorem (Wiener-Hopf factorization)

Let L be a Lévy process that satisfies assumption **(EM)** (and is not a compound Poisson process). The bilateral Laplace transform of \bar{L}_θ at an independent and exponentially distributed time $\theta, \theta \sim \text{Exp}(q)$, with $q > \alpha^*(M)$, can be identified from the Wiener-Hopf factorization of L via

$$E[e^{-\beta L_\theta}] = \int_0^\infty q E[e^{-\beta \bar{L}_t}] e^{-qt} dt = \frac{\kappa(q, 0)}{\kappa(q, \beta)} \quad (5.9)$$

for $\beta \in \{\beta \in \mathbb{C} : \Re(\beta) > -M\}$. The Laplace exponent of the ascending ladder process $\kappa(\alpha, \beta)$, for $\alpha \geq \alpha^*(M)$ and $k > 0$, has an analytic extension to $\beta \in \{\beta \in \mathbb{C} : \Re(\beta) > -M\}$ and is given by

$$\kappa(\alpha, \beta) = k \exp\left(\int_0^\infty \int_0^\infty (e^{-t} - e^{-\alpha t - \beta x}) \frac{1}{t} P_{L_t}(dx) dt\right). \quad (5.11)$$

Infinum/Supremum Processes (continued)

Theorem

Let L be a Lévy process that satisfies assumption **(EM)** (and is not a compound Poisson process). The bilateral Laplace transform of \bar{L}_t and \underline{L}_t at a fixed time $t \in [0, T]$, is given by

$$E[e^{-\beta \bar{L}_t}] = \lim_{A \rightarrow \infty} \frac{1}{2\pi} \int_{-A}^A \frac{e^{t(Y+iv)} \kappa(Y+iv, 0)}{Y+iv} \frac{\kappa(Y+iv, \beta)}{\kappa(Y+iv, \beta)} dv, \quad (5.17)$$

and

$$E[e^{\beta \underline{L}_t}] = \lim_{A \rightarrow \infty} \frac{1}{2\pi} \int_{-A}^A \frac{e^{t(\tilde{Y}+iv)} \bar{\kappa}(\tilde{Y}+iv, 0)}{\tilde{Y}+iv} \frac{\bar{\kappa}(\tilde{Y}+iv, -\beta)}{\bar{\kappa}(\tilde{Y}+iv, -\beta)} dv, \quad (5.18)$$

for $\beta \in \mathbb{C}$ with $\Re \beta \in (-M, \infty)$ and $Y, \tilde{Y} > \alpha^*(M)$.

Payoff Examples

Example (Call option)

The payoff of the standard call option with strike $K \in \mathbb{R}_{\geq 0}$ is $f(x) = (e^x - K)^+$. Let $z \in \mathbb{C}$ with $\Im z \in (1, \infty)$, then the Fourier transform of the payoff function of the call option is

$$\hat{f}(z) = \frac{K^{1+iz}}{iz(1+iz)} \quad (6.1)$$

For $R \in (1, \infty)$ that $g \in L^1_{bc}(\mathbb{R}) \cap L^2(\mathbb{R})$. The weak derivative of g is

$$\partial g(x) = \begin{cases} 0, & \text{if } x < \ln K \\ e^{-Rx}(e^x - Re^x + RK), & \text{if } x > \ln K \end{cases}$$

Payoff Examples (continued)

Example (Digital option)

The payoff of a digital call option with barrier $B \in \mathbb{R} \geq 0$ is $1_{\{e^x > B\}}$. Let $z \in \mathbb{C}$ with $\Im z \in (0, \infty)$, then the Fourier transform of the payoff function of the digital call option is

$$\hat{f}(z) = -\frac{B^{iz}}{iz}. \quad (6.4)$$

Example (Asset-or-nothing digital call)

The payoff of the asset-or-nothing digital call option with barrier $B \in \mathbb{R} \geq 0$ is $f(x) = e^x 1_{\{e^x > B\}}$, and the Fourier transform, for $z \in \mathbb{C}$ with $\Im z \in (1, \infty)$, is

$$\hat{f}(z) = -\frac{B^{1+iz}}{1+iz}. \quad (6.6)$$

Payoff Examples (continued)

Example (Double digital option)

The payoff of the double digital call option with barriers $\underline{B}, \bar{B} \in \mathbb{R} \geq 0$ is $1_{\{\underline{B} < e^x < \bar{B}\}}$. Let $z \in \mathbb{C}$ with $\Im z \in \mathbb{R} \setminus \{0\}$, then the Fourier transform of the payoff function is

$$\hat{f}(z) = \frac{1}{iz} (\bar{B}^{iz} - \underline{B}^{iz}). \quad (6.8)$$

Example (Self-quanto option)

The payoff of a self-quanto call option with strike $K \in \mathbb{R} \geq 0$ is $f(x) = e^x(e^x - K)^+$. Let $z \in \mathbb{C}$ with $\Im z \in (2, \infty)$, then the Fourier transform of the payoff function of the self-quanto call option is

$$\hat{f}(z) = \frac{K^{2+iz}}{(1+iz)(2+iz)} \quad (6.9)$$

Payoff Examples (continued)

Example (Power option)

The payoff of a power call option with strike $K \in \mathbb{R} \geq 0$ and power 2 is $f(x) = [(e^x - K)^+]^2$. Let $z \in \mathbb{C}$ with $\Im z \in (2, \infty)$, then the Fourier transform of the payoff function of the power call option is

$$\hat{f}(z) = \frac{-2K^{2+iz}}{iz(1+iz)(2+iz)} . \quad (6.11)$$

Payoff Examples (continued)

Example (Option on the minimum)

The payoff function of a call option on the minimum of d assets is

$$f(x) = (e^{x_1} \wedge \cdots \wedge e^{x_d} - K)^+,$$

for $x \in \mathbb{R}^d$. The Fourier transform of this payoff function is

$$\hat{f}(z) = \frac{-K^{1+i\sum_{k=1}^d z_k}}{(-1)^d \prod_{k=1}^d (iz_k)(1+i\sum_{k=1}^d z_k)} \quad (6.13)$$

where $z \in \mathbb{C}^d$ with $\Im z_k > 0$ for $1 \leq k \leq d$ and $\Im \left(\sum_{k=1}^d z_k \right) > 1$.

Payoff Examples (continued)

Example (products of single asset payoff functions)

These payoff functions have the form, for $x \in \mathbb{R}^d$,

$$f(x) = \prod_{i=1}^d f_i(x_i),$$

The Fourier transform of these payoff functions is simply the product of the Fourier transform of the 'marginal' payoff functions, since

$$\hat{f}(z) = \int_{\mathbb{R}^d} e^{i\langle z, x \rangle} \prod_{i=1}^d f_i(x_i) dx = \prod_{i=1}^d \int_{\mathbb{R}} e^{iz_i x_i} f_i(x_i) dx_i = \prod_i \hat{f}_{i=1}^d(z_i),$$

for $z \in \mathbb{C}^d$ and $z_i \in \mathbb{C}$, with $\Im z$ in an appropriate range such that $\hat{f} \in L^1(\mathbb{R}^d)$.

Payoff Examples (continued)

Remark

Further examples of multiple asset payoff functions, such as basket and spread options, and their Fourier transforms can be found in Hubalek and Kallsen (2005).

Payoff Examples (continued)

A fixed strike lookback call option with payoff

$$(\bar{S}_T - K)^+ = (S_0 e^{\bar{L}_T} - K)^+ \quad (7.10)$$

Theorem

Let L be a Lévy process that satisfies Assumption (EM). The price of a fixed strike lookback call option with payoff (7.10) is given by

$$\mathbb{C}_T(\bar{S}; K) = \frac{1}{2\pi} \int_{\mathbb{R}} S_0^{R-iu} \varphi_{\bar{L}_T}(-u - iR) \frac{K^{1+iu-R}}{(iu - R)(1 + iu - R)} du,$$

where

$$\varphi_{\bar{L}_T}(-u - iR) = \lim_{A \rightarrow \infty} \frac{1}{2\pi} \int_{-A}^A \frac{e^{T(Y+iv)}}{Y + iv} \frac{\kappa(Y + iv, 0)}{\kappa(Y + iv, iu - R)} dv,$$

for $R \in (1, M)$ and $Y > \alpha^*(M)$.

Payoff Examples (continued)

A one-touch call option with payoff

$$1_{\{\bar{S}_T > B\}} = 1_{\{\bar{L}_T > \log(B/S_0)\}}$$

can be valued as a digital call option where the driving process is the supremum of the underlying Lévy process.

Theorem

Let L be a Lévy process with infinite variation, or with infinite activity and regular upwards, that satisfies Assumption (EM). The price of a one-touch option is given by

$$\begin{aligned} \mathbb{D}C_T(\bar{S}; B) &= \lim_{A \rightarrow \infty} \frac{1}{2\pi} \int_{-A}^A S_0^{R+iu} \varphi_{\bar{L}_T}(u - iR) \frac{B^{-R-iu}}{R + iu} du \quad (7.14) \\ &= P(\bar{L}_T > \log(B/S_0)), \end{aligned}$$

for $R \in (0, M)$ and $Y > \alpha^*(M)$.

Summary

Remark 7.5. Summarizing, when dealing with a *continuous* payoff function, the developed valuation formulae can be applied to *all* Lévy processes; when dealing with a *discontinuous* payoff function, the valuation formulae can handle every Lévy process, *apart* from *compound Poisson* type processes without diffusion component, and *non-regular upwards*, finite variation, Lévy processes. The statement holds for non-path-dependent as well as for *path-dependent exotic* options.

Equity Default Swaps

The protection buyer pays a fixed premium in exchange for an insurance payment in case of 'default'. Let us denote by τ_B the first passage time below the barrier level B , i.e.

$$\tau_B = \inf\{t \geq 0; S_t \leq B\}.$$

The protection buyer pays a fixed premium denoted by \mathcal{K} at the dates $T_1, T_2, \dots, T_N = T$, provided that default has not occurred, i.e. $T_i < \tau_B$.

Equity Default Swaps

In case of default, the protection seller makes the insurance payment C , which is typically 50% of the initial value. Now, the premium he is fixed such that the value of the equity default swap at inception is zero, hence we get

$$\mathcal{K} = \frac{CE[e^{-r\tau_B}1_{\{\tau_B \leq T\}}]}{\sum_{i=1}^N E[e^{-rT_i}1_{\{\tau_B > T_i\}}]} \quad (7.16)$$

where r denotes the risk-free interest rate.

Remark 7.7. Asmussen, Madan, and Pistorius (2007) study the valuation of equity default swaps under an approximation of the CGMY model and present some empirical results from calibration to market data.