Option Pricing for GARCH Models with Markov Switching

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Joint work with

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We set out to provide a generalization of the GARCH model of volatility and its application to the pricing of option. The generalization involves having the dynamics of the GARCH process change over time according to one of N regimes. It is well known that the GARCH model has been a very successful innovation for modeling dynamics of volatility. We generalize it to capture the business cycles.

Motivation:

The creation of the Black-Scholes formula (1973) is one of the most celebrated accomplishments in the recent economic history. However, since its introduction it has been discovered that it has certain biases. The motivation of Markov Switching GARCH Models is to correct some of these known biases, specifically:

1. Return Skewness and Leptokurtic behaviour.
2. Volatility Smile.
3. Volatility Persistence.

Further advantages of our model

1. it can incorporate structural changes in the volatility dynamics due to the changes in economic conditions in order to capture the effects of business cycles.
2. it can provide flexibility in describing the volatility persistence of the shocks by introducing an extra source of volatility persistence driven by the Markov process.
3. Some empirical studies mentioned that Markov switching GARCH models can improve the performance of GARCH models in their abilities to forecast volatilities of various asset classes, for instance, foreign exchange rates, (Klaassen (2002))

Let X be a Markov chain with 2 states: The chain models the 'good' and 'bad' states of the economy. We identify 'good' with the vector $(1,0)'$ and 'bad' with the vector $(0,1)'$.

We consider the probabilities of switching from ‘good’ to ‘bad’ or ‘good’ to ‘good’ etc.

Write

$$\pi_{ji} = \mathcal{P}(X_t = e_j | X_{t-1} = e_i) \ , \tag{2.1}$$

and $\Pi = (\pi_{ji})$, $1 \leq i, j \leq 2$.

The idea of the model

A GARCH series H models the variance of returns by:

$$H_t^2 = \alpha_0 + \alpha_1(\nu_{t-1} - \rho H_{t-1})^2 + \beta H_{t-1}^2,$$

where ν is the 'innovation' in the return process. (That is, the difference between the observed noise and expected noise.)

In our model the α_0, α_1 and β depend on the state of the economy X.

$$H_t^2(X_{t-1}) = <\alpha_0, X_{t-1}> + <\alpha_1, X_{t-1}> (\nu_{t-1} - \rho H_{t-1})^2 + <\beta, X_{t-1}> H_{t-1}^2 , \quad (2)$$

where $< x, y >$ denotes the scalar product of two vectors $x, y \in \mathcal{R}^2$.

We also suppose the market interest rate $\{r(X_t)\}_{t \in \mathcal{T}}$ of the bank account B depends on the state of the economy and is modelled as:

$$r_t =< r, X_t > \ , \tag{3}$$

where $r = (r_1, r_2)$. The unit risk premium $\{\lambda(X_t)\}_{t \in \mathcal{T}}$ of the risky asset S also depends on the state of the economy and is modelled as:

$$\lambda_t =< \lambda, X_t > \ , \tag{4}$$

where $\lambda = (\lambda_1, \lambda_2)$.

Asset price dynamics

Suppose then the dynamics of the bond and the stock price are given by

$$\begin{aligned} B_t &= B_{t-1} \exp(r_{t-1}) \ , \quad B_0 = 1 \ , \\ S_t &= S_{t-1} \exp \left[r_{t-1} + \lambda_{t-1} H_t^2 + H_t \nu_t \right] \ , \quad S_0 = s. \end{aligned}$$

Write the log-return as

$$\begin{aligned} Y_t &= \ln\left(\frac{S_t}{S_{t-1}}\right) \\ &= r_{t-1} + \lambda_{t-1} H_t^2 + H_t \nu_t \ . \end{aligned} \tag{5}$$

For $t \in \{1, 2, ..., T\}$, let $M_Y(t, u)$ be the conditional moment generating function of Y_t given the history of X and Y to time $t-1$ under $\mathcal{P}$. Since $Y_t|\mathcal{F}_{t-1} \sim N(r_{t-1} + \lambda_{t-1}H_t^2, H_t^2)$,

$$\begin{aligned} M_Y(t, u) &= E\Big[\exp(uY_t)|\mathcal{F}_{t-1}\Big] \\ &= \exp\left[(r_{t-1} + \lambda_{t-1}H_t^2)u + \frac{1}{2}u^2H_t^2\right]. \end{aligned}$$

Markov switching conditional Esscher transform

The Esscher transform was first used in actuarial science. It defines a new probability measure. We shall define a Markov switching conditional Esscher transform.

As in Bühlmann et al. (1996), for a predictable sequence of real random variables $\theta = \{\theta_t\}$, define

$$\begin{aligned} \Lambda_t &= \prod_{k=1}^{t} \frac{e^{\theta_k Y_k}}{M_Y(k, \theta_k)} \\ &= \exp\left\{ \sum_{k=1}^{t} \left[\theta_k Y_k - (r_{k-1} + \lambda_{k-1} H_k^2)\theta_k - \frac{1}{2}\theta_k^2 H_k^2 \right] \right\}. \end{aligned}$$

Then the Esscher transform is obtained if we define $\mathcal{P}^\theta$ by setting

$$\frac{d\mathcal{P}^\theta}{d\mathcal{P}}|_{\mathcal{F}_t} = \Lambda_t^\theta.$$

Write $\mathcal{F}_{t-1}$ for the history of X and Y up to time $t-1$.

Let $M_Y(u,t;\theta_t)$ denote the conditional moment generating function of Y_t given $\mathcal{F}_{t-1}$ under $\mathcal{P}^\theta$. Then, we have the following lemma.

Lemma:

$$\begin{aligned} & M_Y(u,t;\theta_t) \\ = \; & E^\theta[e^{uY_t}|\mathcal{F}_{t-1}] = \exp\left[r_{t-1}u + \left(\lambda_{t-1} + \theta_t + \frac{1}{2}u\right)H_t^2 u\right]. \end{aligned}$$

We shall take the Esscher transform parameter:

$$\theta_t = - < \lambda, X_{t-1} > -\frac{1}{2} \ . \tag{6}$$

Then, the discounted price process $\{\frac{S_t}{B_t}\}_{t \in \mathcal{T}}$ is a martingale given $\{\mathcal{F}_t\}_{t \in \mathcal{T}}$ under $\mathcal{P}^{\theta}$.

Consider the accumulated returns from $T-k+1$ to T:

$$A_{k,T} = Y_{T-k+1} + Y_{T-k+2} + ... + Y_{T-1} + Y_T$$

Then we can find a recursion which gives the characteristic function of $A_{k,T}$:

$$\begin{aligned}
& \Phi_{A_{T-k,T}|X,T-k}(u) \\
= \; & E^{\theta}\left[\exp\left(iu(A_{k,T}) \right) \middle| \mathcal{F}_{T-k} \right] \\
= \; & \exp\Big[F_k(u, \alpha_{0,T-k+1}, \alpha_{T-k+1}, \beta_{T-k+1}, r_{T-k}, \rho^{\theta}_{T-k+1}) + \\
& G_k(u, \alpha_{0,T-k+1}, \alpha_{T-k+1}, \beta_{T-k+1}, r_{T-k}, \rho^{\theta}_{T-k+1}) H^2_{T-k+1} \Big],
\end{aligned}$$

where explicit forms are found for F_k and G_k.

An Option Pricing Formula

The characteristic function is the Fourier transform of the density. Therefore, we can work in Fourier transform space to estimate option prices. Then, at any time $t \in \mathcal{T}$, the price of a European contingent claim with payoff $V(S_T)$ is:

$$V(t, T, S_t|\mathcal{F}_t) = E^{\theta}\left[\text{exp}\left(-\sum_{k=t}^{T} r_k\right)V(S_T)\middle|\mathcal{F}_t\right].$$

The option price equals

$$\int \Phi_{A_{t,T}|X,t}(u)\hat{V}(u)du \; . \qquad (1)$$

Here $\hat{V}(u)$ is the Fourier transform of $V(S_T)$.

The final expression looks like:

Let $C(t, T, \tilde{s}_t|\mathcal{F}_t)$ denote the time-t price of a European call option with strike price K and maturity at time T . Then

$$
\begin{aligned}
& C(t, T, \tilde{s}_t|\mathcal{F}_t) \\
= \; & \frac{1}{2}e^{\tilde{s}_t} + \frac{1}{\pi}\exp\Big(- \sum_{k=t}^{T} < r, X_k > \Big) \\
& \int_0^\infty Re\Big[\frac{K^{-iu}\Phi^*_{A_{t,T}|X,t}(i-u)}{iu}\Big]du \\
- \; & K\exp\Big(- \sum_{k=t}^{T} < r, X_k > \Big) \\
& \Big\{\frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\Big[\frac{K^{-iu}\Phi^*_{A_{t,T}|X,t}(-u)}{iu}\Big]du\Big\}.
\end{aligned}
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I welcome any comments or questions.