Modeling and Pricing of Variance Swaps for Local Stochastic Volatilities with Delay and Jumps *

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Outline of Presentation

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Introduction: Stock Price

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t), \quad t > 0,$$

where $\mu \in R$ is the mean rate of return, the volatility term $\sigma > 0$ is a continuous and bounded function and $W(t)$ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$. We also let $r > 0$ be the risk-free rate of return of the market. We denote $S_t = S(t - \tau), \quad t > 0$ and the initial data of $S(t)$ is defined by $S(t) = \varphi(t)$, where $\varphi(t)$ is a deterministic function with $t \in [-\tau, 0], \quad \tau > 0$.

Introduction: Stochastic Volatility with Delay

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(u,S_u)dW(u) + \int_{t-\tau}^{t} \sigma(u,S_u)d\tilde{N}(u)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t)$$

where $N(t)$ is a Poisson process independent of $W(t)$ with intensity $\lambda > 0$ and $\tilde{N}(t) := N(t) - \lambda t$.

Introduction: Stochastic Volatility with Delay (cntd)

Our model of stochastic volatility exhibits jumps and also past-dependence: the behavior of a stock price right after a given time t not only depends on the situation at t, but also on the whole past (history) of the process $S(t)$ up to time t. This draws some similarities with fractional Brownian motion models (see Mandelbrot (1997)) due to a long-range dependence property. Another advantage of this model is mean-reversion. This model is also a continuous-time version of GARCH(1,1) model (see Bollerslev (1986)) with jumps.

Motivation: Why Delay?

Some statistical studies of stock prices indicate the dependence on past returns:

- Sheinkman and LeBaron (1989),
- Akgiray (1989)

Motivation: Why Delay? (cntd)

• Kind, Liptser and Runggaldier (1991) obtained a diffusion approximation result for processes satisfying some equations with past-dependent coefficients, and they applied this result to a model of option pricing, in which the underlying asset price volatility depends on the past evolution to obtain a generalized (asymptotic) Black-Scholes formula.

Motivation: Why Delay? (cntd)

• Hobson and Rogers (1998) suggested a new class of nonconstant volatility models, which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model. The volatility is nonconstant and can be regarded as an endogenous factor in the sense that it is defined in terms of the past behavior of the stock price. This is done in such a way that the price and volatility form a multi-dimensional Markov process.

Motivation: Why Delay? (cntd)

- Chang and Yoree (1999) studied the pricing of an European contingent claim for the (B, S)-securities markets with a hereditary price structure in the sense that the rate of change of the unit price of the bond account and rate of change of the stock account S depend not only on the current unit price but also on their historical prices.

Motivation: Why Delay? (cntd)

Clearly related to our work is the work by

- Mohammed, Arriojas and Pap (2001) devoted to the derivation of a delayed Black-Scholes formula for the (B, S)-securities market using PDE approach.

Motivation: Why Delay? (cntd)

Our model of stochastic volatility exhibits past-dependence: the behavior of a stock price right after a given time t not only depends on the situation at t, but also on the whole past (history) of the process $S(t)$ up to time t. This draws some similarities with fractional Brownian motion models due to a long-range dependence property.

Motivation: Why Delay? (cntd)

Our work is also based on the GARCH(1,1) model (see Bollerslev (1986))

$$\sigma_n^2 = \gamma V + \alpha \ln^2(S_{n-1}/S_{n-2}) + (1-\alpha-\gamma)\sigma_{n-1}^2$$

or, more general,

$$\sigma_n^2 = \gamma V + \frac{\alpha}{l} \ln^2(S_{n-1}/S_{n-1-l}) + (1-\alpha-\gamma)\sigma_{n-1}^2$$

and the work of Duan (1995) where he showed that it is possible to use the GARCH model as the basis for an internally consistent option pricing model.

Motivation: Why Delay? (cntd)

If we write down the last equation in differential form we can get the continuos-time GARCH with expectation of log-returns of zero:

$$\frac{d\sigma^2(t)}{dt} = \gamma V + \frac{\alpha}{\tau}\ln^2\left(\frac{S(t)}{S(t-\tau)}\right) - (\alpha+\gamma)\sigma^2(t)$$

If we incorporate non-zero expectation of log-return (using Itô Lemma for $\ln\frac{S(t)}{S(t-\tau)}$) then we arrive to our continuous-time GARCH model for stochastic volatility with delay:

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sigma(s,S_s)dW(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t).$$

Motivation: Why Jumps?

There is currently fairly compelling evidence for jumps in the level of financial prices. The most convincing evidence comes from recent nonparametric work using high-frequency data as in Barndorff-Nielsen and Shephard (2007) and Aït-Sahalia and Jacod (2008) among others. Also, paper by Todorov and Tauchen (2008) conducts a non-parametric analysis of the market volatility dynamics using high-frequency data on the VIX index compiled by the CBOE and the $S\&P500$ index.

Motivation: Why Jumps? (cntd)

The data suggest that stock market volatility is best described as a pure jump process without a continuous component. Their results imply that a plausible model for stochastic volatility is a model of pure-jump type whose driving jumps come from a very active Lévy process. Some attempts have been made to incorporate jumps in stochastic volatility to price variance and volatility swaps (see Howison *et al.* (2004)).

Motivation: Why Jumps? (cntd)

The jumps in stock market volatility are found to be so active that this discredits many recently proposed stochastic volatility models without jumps (see Todorov and Tauchen (2008), Bollerslev *et al* (2008)).

Motivation: Why Jumps? (cntd)

The key risk factors considered in option pricing models, besides the diffusive price risk of the underlying asset, are stochastic volatility and jumps, both in the asset price and its volatility. Models that include some or all of these factors were developed by Merton (1976), Heston (1993), Bates (1996), Bakshi et al. (1997) and Duffie et al. (2000).

Motivation: Why Jumps? (cntd)

The importance of jumps in volatility has become apparent in recent studies, which try to explain the time series properties of both stock and option prices, like Eraker *et al.* (2003), or Broadie *et al.* (2007). In an asset allocation context, the main papers analyzing the impact of jumps are Liu *et al.* (2003), Liu and Pan (2003) and Dieckmann and Gallmeyer (2005). In the presence of jumps markets are incomplete and the analysis far less tractable.

Motivation: Why Jumps? (cntd)

Technical issues aside, jumps are important because they represent a significant source of non-diversifiable risk as discussed at length in Bollerslev *et al.*(2008).

Eraker *et al.*(2000) use returns data to investigate the performance of models with jumps in volatility using the class of jump-in-volatility models proposed by Duffie *et al.* (2000). The results in Eraker, Johannes and Polson (2000) show that the jump-in-volatility models provide a significant better fit to the returns data.

Motivation: Why Jumps? (cntd)

Another advantage of our stochastic volatility model with delay and jumps is mean-reversion: the volatility is allowed to mean revert. Such models have shown some success in modeling interest rate (i.e., Aït-Sahalia (1996)). The sharp decline of option implied spot volatility following the extreme peak caused by the 1987 crash would be indicative of such a model.

The Model

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t), \quad t > 0,$$

$$\begin{aligned} \frac{d\sigma^2(t,S_t)}{dt} &= \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(u, S_u)dW(u) + \int_{t-\tau}^{t} \sigma(u, S_u)d\tilde{N}(u)\right]^2 \\ &- (\alpha + \gamma)\sigma^2(t, S_t) \end{aligned}$$

Conditions

C1) $\sigma(t, S_t)$ satisfies local Lipschitz and growth conditions;

C2) $\int_0^T E\sigma^2(t, S_t)dt < +\infty$;

C3) $\int_0^T (\frac{r-\mu}{\sigma(t,S_t)})^2 dt < +\infty$ a.s.

Condition C1) guarantees the existence and uniqueness of a solution of equations for $S(t)$ and $\sigma^2(t)$ in Section 2 (see Mohammed (1998)). Condition C2) guarantees the existence of Itô integral in eqaution for $\sigma^2(t)$ and C3) guarantees the existence of risk-neutral measure P^* (see below).

Risk-Neutral World

1) There is a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{-\int_0^T \theta(s)dW(s) - \frac{1}{2}\int_0^T \theta^2(s)ds\right\}$$

is its Radon-Nikodym density, where

$$\theta(t) = \frac{\mu - r}{\sigma(t, S_t)}.$$

Risk-Neutral World (cntd)

2) The discounted asset price $D(t)$ is a positive local martingale with respect to $\mathbb{P}^*$, and

$$W^*(t) = \int_0^t \theta(s)ds + W(t) \tag{2}$$

is a standard Brownian motion with respect to $\mathbb{P}^*$.

Risk-Neutral World (cntd)

$$dS(t) = rS(t)dt + \sigma(t, S_t)S(t)dW^*(t) \qquad (3)$$

and the asset volatility is defined then as follows:

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(s, S_s)dW^*(s) + \int_{t-\tau}^{t} \sigma(u, S_u)d\tilde{N}(u) - (\mu - r)\tau\right]^2 - (\alpha + \gamma)\sigma^2(t, S_t). \qquad (4)$$

where $W^*(t)$ is defined in (2).

Variance Swaps

Variance swaps are forward contracts on future realized stock variance, the square of the future volatility. The easy way to trade variance is to use variance swaps, sometimes called realized variance forward contracts (see Carr and Madan (1998)). Although options market participants talk of volatility, it is variance, or volatility squared, that has more fundamental significance (see Demeterfi, K., Derman, E., Kamal, M., and Zou, J. (1999)).

Variance Swaps (cntd)

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$N(\sigma_R^2(S) - K_{var}),$$

where $\sigma_R^2(S)$ is the realized stock variance(quoted in annual terms) over the life of the contract,

Variance Swaps (cntd)

$$\sigma_R^2(S) := \frac{1}{T}\int_0^T \sigma^2(s)ds,$$

K_{var} is the delivery price for variance, and N is the notional amount of the swap in dollars per annualized volatility point squared. The holder of variance swap at expiration receives N dollars for every point by which the stock's realized variance $\sigma_R^2(S)$ has exceeded the variance delivery price K_{var}. We note that usually $N = \alpha I$, where α is a converting parameter such as 1 per volatility-square, and I is a long-short index (+1 for long and -1 for short).

Variance Swaps (cntd)

Valuing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract P on future realized variance with strike price K_{var} is the expected present value of the future payoff in the risk-neutral world:

$$\mathcal{P}^* = E_{P^*}\{e^{-rT}(\sigma_R^2(S) - K_{var})\},$$

where r is the risk-free discount rate corresponding to the expiration date T, and E_{P^*} denotes the expectation under the risk-neutral measure P^*.

Variance Swaps (cntd)

In tis way, *a variance swap for stochastic volatility with delay* is a forward contract on annualized variance $\sigma^2_R(t, S_t)$. Its payoff at expiration equals to

$$N(\sigma^2_R(S) - K_{var}),$$

where $\sigma^2_R(S)$ is the realized stock variance(quoted in annual terms) over the life of the contract,

$$\sigma^2_R(S) := \frac{1}{T}\int_0^T \sigma^2(u, S(u-\tau))du, \quad \tau > 0.$$

Pricing of Variance Swaps (cntd)

Let us take the expectations under risk-neutral measure $\mathbb{P}^*$ on the both sides of the equation (4) above. Denoting $v(t) = \mathbb{E}^*[\sigma^2(t, S_t)]$, we obtain the following deterministic delay differential equation:

$$\frac{dv(t)}{dt} = \gamma V + \alpha\tau(\mu - r)^2 + \frac{\alpha(1+\lambda)}{\tau}\int_{t-\tau}^{t} v(s)ds - (\alpha+\gamma)v(t). \quad (5)$$

Pricing of Variance Swaps (cntd)

Notice that (5) has a stationary solution

$$v(t) \equiv X = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}. \tag{6}$$

Pricing of Variance Swaps (cntd)

Hence, the expectation of the realized variance, or say the fair delivery price K_{var} of *a variance swap for stochastic volatility with delay in stationary regime under risk-neutral measure* $\mathbb{P}^*$ equals to

$$\begin{aligned} K_{var} &= \mathbb{E}^*[v] = \frac{1}{T}\int_0^T v(t)dt \\ &= \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}. \end{aligned} \quad (7)$$

Pricing of Variance Swaps (cntd)

The price P of a variance swap at time t given delivery price K in this case should be:

$$P = e^{-r(T-t)}[\frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda} - K].$$

Pricing of Variance Swaps (cntd)

In general case, there is no way to write a solution of (5) in explicit form for arbitrarily given initial data. But we can write an approximate solution for $v(t)$ when t has large values:

$$v(t) \approx X + Ce^{-\gamma t} = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda} + Ce^{(\alpha\lambda - \gamma)t}. \qquad (8)$$

where

$$C = v(0) - X = \sigma_0^2 - \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}. \qquad (9)$$

Pricing of Variance Swaps (cntd)

We note, that the characteristic equation for the equation (5) in this case has the following look

$$\rho^2 + \rho(\gamma - \alpha\lambda) = 0$$

and the solution of the equation is

$$\rho = (\alpha\lambda - \gamma).$$

Pricing of Variance Swaps (cntd)

Hence, the expectation of the realized variance, or say the fair delivery price K_{var} of *variance swap for stochastic volatility with delay in general case under risk-neutral measure* $\mathbb{P}^*$ equals to

$$\begin{aligned} K_{var} &= \mathbb{E}^*[v] = \frac{1}{T}\int_0^T v(t)dt \\ &\approx \frac{1}{T}\int_0^T [V + \alpha\tau(\mu - r)^2/\gamma + (\sigma_0^2 - V - \alpha\tau(\mu - r)^2/\gamma)e^{(\alpha\lambda-\gamma)t}]dt \\ &= \frac{\gamma V + \alpha\tau(\mu-r)^2}{\gamma-\alpha\lambda} + (\sigma_0^2 - \frac{\gamma V + \alpha\tau(\mu-r)^2}{\gamma-\alpha\lambda})\frac{e^{(\alpha\lambda-\gamma)T}-1}{T(\alpha\lambda-\gamma)}. \end{aligned} \tag{10}$$

Pricing of Variance Swaps (cntd)

the price P of a variance swap at time t given delivery price K in this case should be:

$$P \approx e^{-r(T-t)}[\frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda} + (\sigma_0^2 - \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda})\frac{e^{(\alpha\lambda-\gamma)T} - 1}{T(\alpha\lambda - \gamma)} - K].$$

Delay as a Measure of Risk

$$\begin{aligned} K_{var} &= \mathbb{E}^*[v] = \frac{1}{T}\int_0^T v(t)dt \\ &\approx \frac{1}{T}\int_0^T [V + \alpha\tau(\mu - r)^2/\gamma + (\sigma_0^2 - V - \alpha\tau(\mu - r)^2/\gamma)e^{(\alpha\lambda-\gamma)t}]dt \\ &= \frac{\gamma V + \alpha\tau(\mu-r)^2}{\gamma - \alpha\lambda} + (\sigma_0^2 - \frac{\gamma V + \alpha\tau(\mu-r)^2}{\gamma-\alpha\lambda})\frac{e^{(\alpha\lambda-\gamma)T}-1}{T(\alpha\lambda-\gamma)}. \end{aligned} \tag{11}$$

This expression contains all the information about our model, since it contains all the initial parameters (we note that $\gamma > \alpha\lambda$).

We note that $\sigma_0^2 = \sigma^2(0, \varphi(-\tau))$. So the sign of the second term in (11) depends on the relationship between σ_0^2 and $\frac{\gamma V + \alpha\tau(\mu-r)^2}{\gamma-\alpha\lambda}$.

Delay as a Measure of Risk

If $\sigma_0^2 > \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}$, the second term in (11) is positive and $\mathbb{E}^*[v]$ stays above $\frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}$, which means the risk is high.

If $\sigma_0^2 < \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}$, the second term in (11) is negative and $\mathbb{E}^*[v]$ stays below $\frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda}$, which means the risk is low.

Delay as a Measure of Risk

Therefore,

$$\sigma_0^2 = \frac{\gamma V + \alpha\tau(\mu - r)^2}{\gamma - \alpha\lambda} \tag{12}$$

defines the measure of risk in the stochastic volatility model with delay and jumps.

To reduce the risk we need to take into account the following relationship with respect to the delay τ (which follows from (12)):

$$\tau > \frac{\sigma_0^2(\gamma - \alpha\lambda) - \gamma V}{\alpha(\mu - r)^2}. \tag{13}$$

In this case, there is a way to control the delay.

Numerical Example: $S\&P60$ Canada Index

Statistics on Log Returns $S\&P60$ Canada Index	
Series:	Log Returns $S\&P60$ Canada Index
Sample:	1 1300
Observations:	1300
Mean	0.000235
Median	0.000593
Maximum	0.051983
Minimum	-0.101108
Std. Dev.	0.013567
Skewness	-0.665741
Kurtosis	7.787327

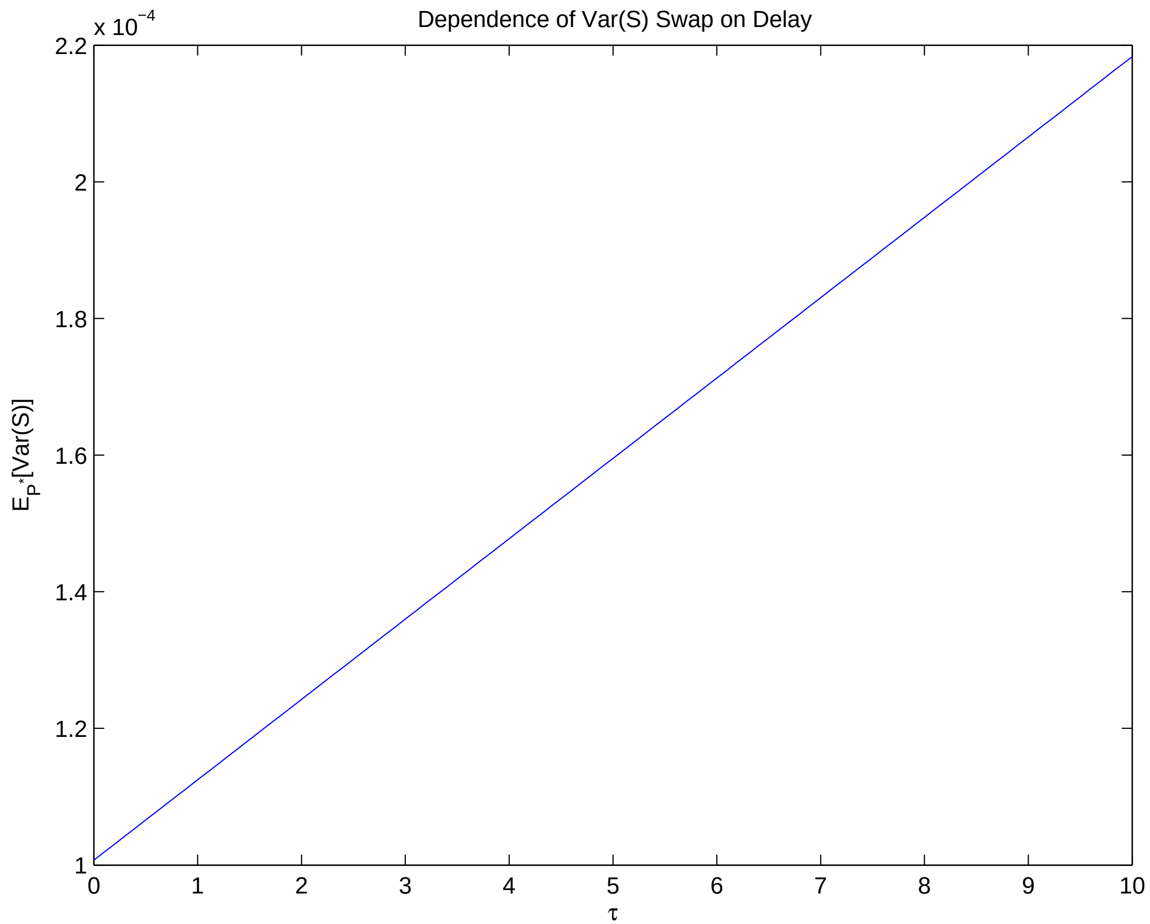

Fig. 1. Dependence of Variance Swap with Delay on Delay ($S\&P60$ Canada Index).

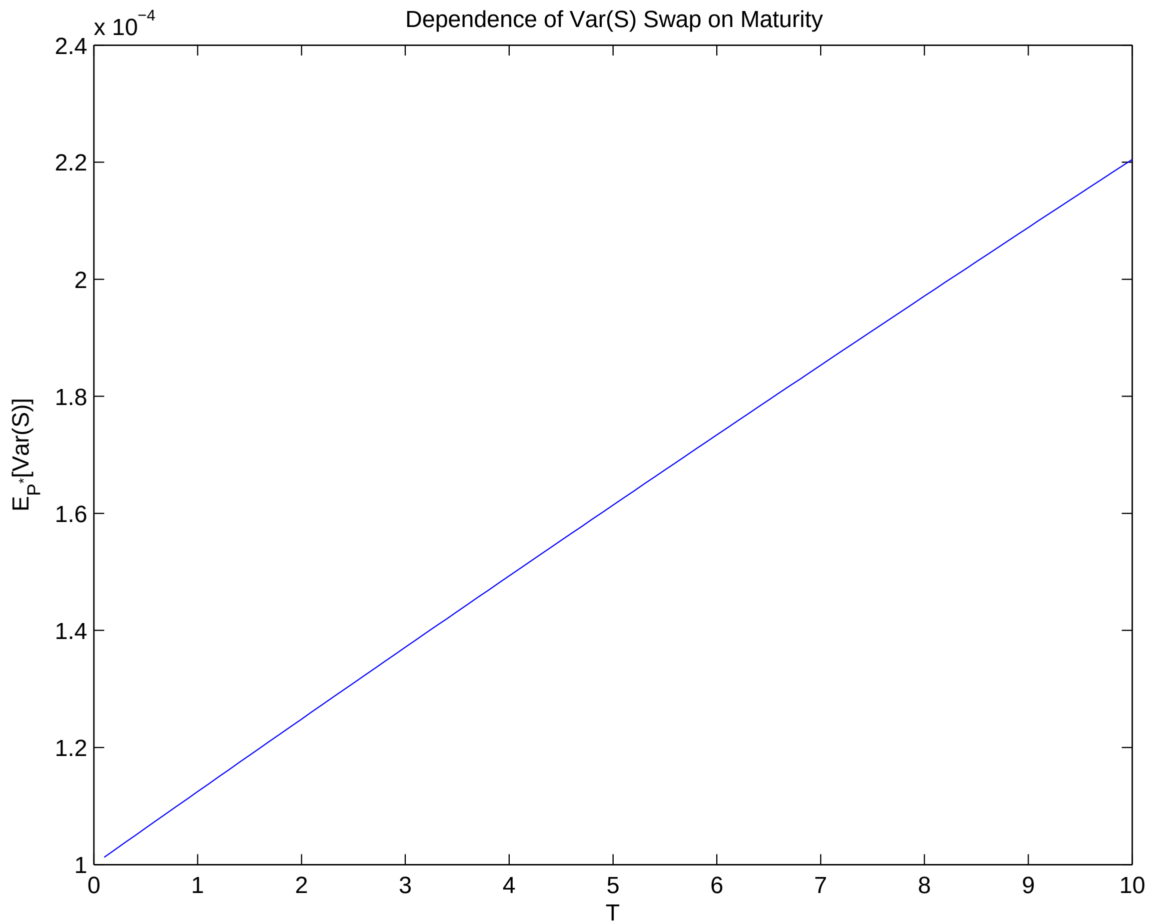

Fig. 2. Dependence of Variance Swap with Delay on Maturity ($S\&P60$ Canada Index).

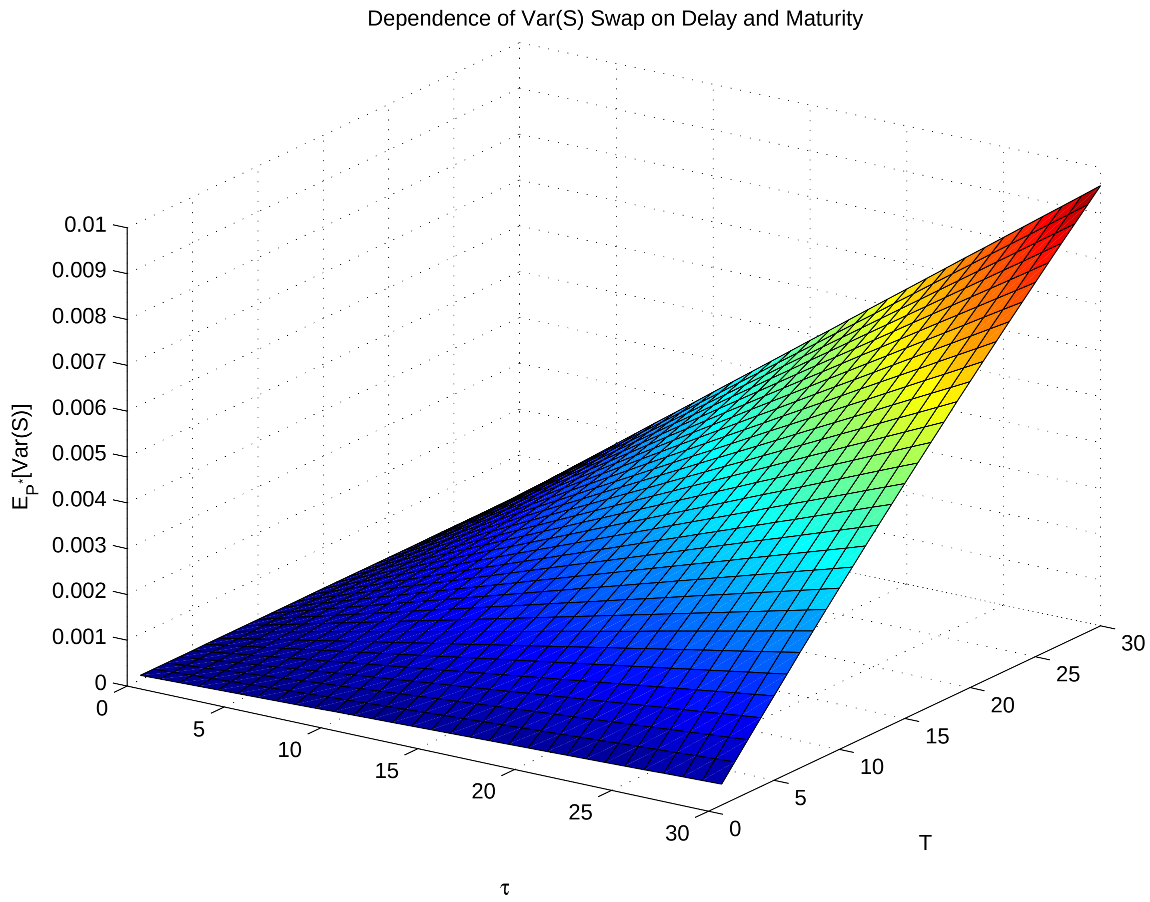

Fig. 3. Variance Swap with Delay for $S\&P60$ Canada Index.

Numerical Example 2: $S\&P500$ Index

Statistics on Log Returns $S\&P500$ Index	
Series:	Log returns $S\&P500$ Index
Sample:	1 1006
Observations:	1006
Mean	0.000263014
Median	8.84424E-05
Maximum	0.034025839
Minimum	-0.045371484
Std. Dev.	0.00796645
Variance	6.34643E-05
Skewness	-0.178481359
Kurtosis	3.296144083

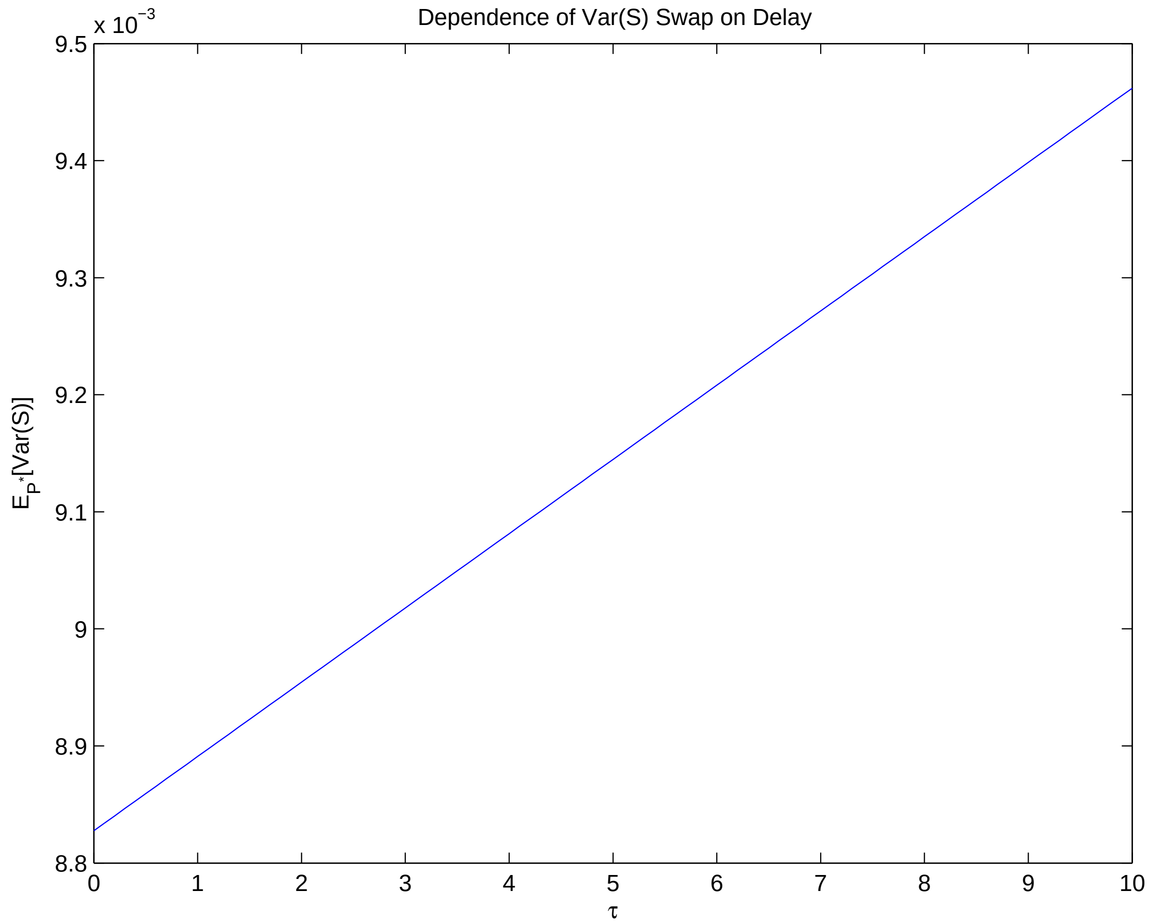

Fig. 4. Dependence of Variance Swap with Delay on Delay ($S\&P500$ Index).

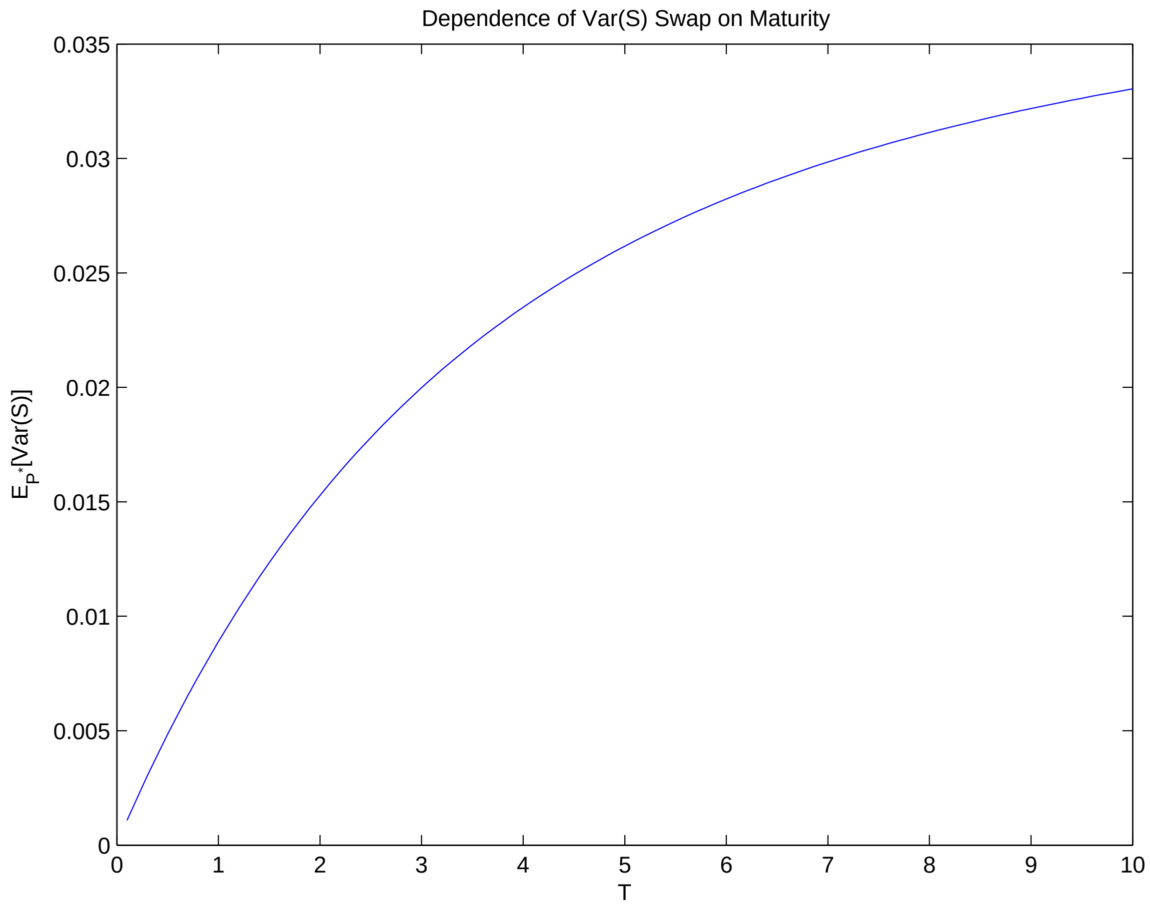

Fig. 5. Dependence of Variance Swap with Delay on Maturity ($S\&P$500 Index).

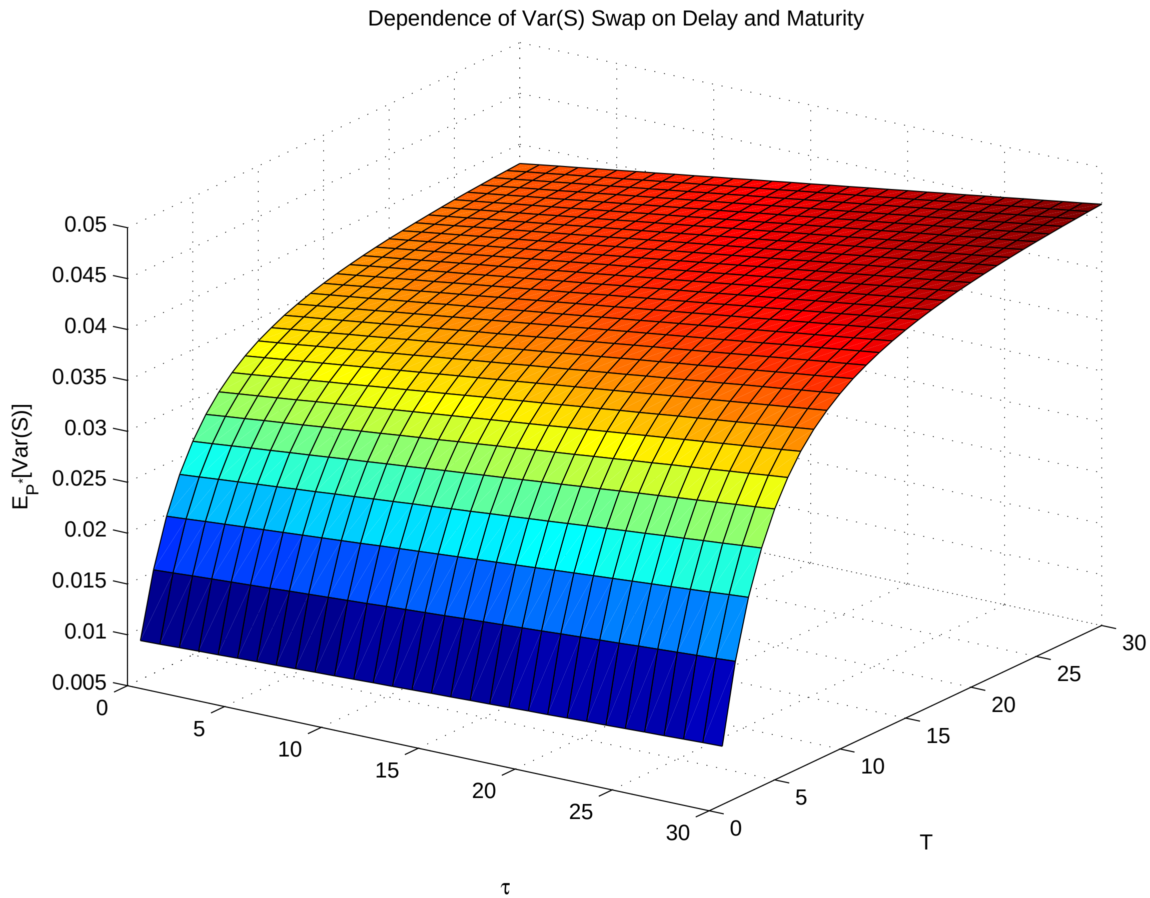

Fig. 6. Variance Swap with Delay for $S\&P500$ Index.

Students' Projects

1. **Kevin Malenfant**: *Stock Price:*

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t), \quad t > 0,$$

Lévy-based Stochastic Volatility with Delay:

$$\begin{aligned}\frac{d\sigma^2(t,S_t)}{dt} = &\gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(u, S_u)dL(u)\right]^2 \\ &- (\alpha+\gamma)\sigma^2(t, S_t)\end{aligned}$$

where $L(t)$ is a Lévy process.

Students' Projects

2. **Matthew Couch**: *Stock Price:*

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t) + dJ_t, \qquad t > 0,$$

where J_t is a compound Poisson process.

Stochastic Volatility $\sigma(t, S_t)$ *with Delay:*

$$\frac{d\sigma^2(t,S_t)}{dt} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(u, S_u)dW(u)\right]^2 - (\alpha + \gamma)\sigma^2(t, S_t).$$

Students' Projects

3. **A Prospective Student:** *Stock Price:*

$$dS(t) = \mu S(t)dt + \sigma(t, S_t)S(t)dW(t) + dL_1(t), \quad t > 0,$$

where $L_1(t)$ is a Lévy process.

Stochastic Volatility $\sigma(t, S_t)$ *with Delay:*

$$\begin{aligned}\frac{d\sigma^2(t,S_t)}{dt} = &\gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(u, S_u)dL_2(u)\right]^2 \\ &- (\alpha + \gamma)\sigma^2(t, S_t),\end{aligned}$$

where $L_2(t)$ is another Lévy process independent (or correlated) of (with) $L_1(t)$.

The End

Thank you very much for your attentiona and time!