Chapter 6: Modelling Forwards and Swaps Using the Heath-Jarrow-Morton (HJM) Approach *

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'Lunch at the Lab' Talk
January 21st, 2010

*Book Review: 'Stochastic Modeling of Electricity and Related Markets' by F. Benth, J. Benth & S. Koekebakker, 2008, World Sci. Publ.

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Intro

In the fixed income markets, instead of modelling the prices via one- or multi-factor spot rate models, the dynamics of the forward rates are directly specified.

This approach leads to simple evaluations of bond prices through integration in time, and known as the HJM approach (see Heath, Jarrow and Morton (1992)).

The HJM approach has later been adopted to modelling forward and futures prices in commodity markets, and this will be the topic of the current Chapter 6.

Intro

Most commodity markets trade in forward contracts with settlement at a fixed time. In this case the adoption of the HJM approach is rather starightforward, and we start this Chapter 6 with a thorough discussion of such contracts.

However, in electricity, gas or weather markets, the commodity is delivered over a period, and it is no longer obvious how to apply the HJM approach.

We describe the approach for electricity and gas contracts, which we commonly denote as swaps.

Intro

As we will see, the strightforward implementation of HJM for swaps leads to intractable models. The alternative is to model only those contracts which are traded.

This resembles closely what is known as swap and LIBOR models in the interest rate markets (see Brigo and Mercurio (2001)). These models are also known as market models.

Intro

One may also generate models for swap prices by integrating the forward price over the delivery period. However, forward prices are not directly observed in the electricity or gas markets, which introduces some problems with estimation.

One may derive data by smoothing the swap curve, however, this may be a dubious path to follow since the data may depend on the algorithm chosen.

In this Chapter 6 we concentrate on the theoretical foundation for directmodelling of swaps and forwards.

The HJM Modelling Idea for Forward Contracts

In fixed income markets, the HJM approach models the forward rates directly, and frequently a GBM model is used.

The direct analogy to energy markets would be to let the forward price dynamics in the risk-neutral setting be given as

$$\frac{df(t,\tau)}{f(t,\tau)} = \sigma(t,\tau)dW(t),$$

where W is a standard Brownian motion. The function σ, modelling the volatility term structure in the market, is assumed to be positive. Usually, this term structure is supposed to be continuous in both current time t and time to delivery $\tau, 0 \leq t \leq \tau \leq \mathcal{T}$.

The HJM Modelling Idea for Forward Contracts

The market has a maximum time of delivery given by $\mathcal{T}$. The explicit dynamics of the forward is

$$f(t,\tau) = f(0,\tau)\exp(-\frac{1}{2}\int_0^t \sigma^2(u,\tau)du + \int_0^t \sigma(u,\tau)dW(u),$$

withy $f(0,\tau)$ being the initial forward curve observed today in the market. Thus, the forward price will have independent and normally distributed logreturns under the risk-neutral measure.

Note that since it is costless to enter a forward contract, it has zero expected return. Thus, the dynamics is without drift.

The HJM Modelling Idea for Forward Contracts

Commodities like gas and electricity have frequently large jumps in the spot price, which in theory should be reflected in the forward price. Hence, when creating forward price models directly, it is natural to include jump processes like we did for the spot price dynamics. One possibility is to state the dynamics in differential form

$$\frac{df(t,\tau)}{f(t,\tau)} = \sigma(t,\tau)dW(t) + \eta(t,\tau)dJ(t)$$

under the risk-neutral measure. Here, J is an II process. However, with this dynamics we may obtain negative forward prices, and moreover, the explicit representation becomes highly complicated.

The HJM Modelling Idea for Forward Contracts

By the Itô Formula, the explicit dynamics becomes

$$\begin{aligned} f(t,\tau) &= f(0,\tau)\exp\left(\int_0^t [a(s,\tau)-\tfrac{1}{2}\sigma^2(s,\tau)]ds + \int_0^t \sigma(s,\tau)dW(s)\right) \\ &\times \prod_{s\le t}(1+\eta(s,\tau)\Delta J(s))\exp(-\eta(s,\tau)\Delta J(s)). \end{aligned}$$

We get negative forward prices whenever jumps of magnitude smaller than -1 is allowed in the II process. Thus, to ensure positive forward prices, we need to assume that $\Delta J(s) > -1$, which is equivalent to saying that the compensator measure $l(dz, ds)$ is supported on the interval $z \in (-1, +\infty)$.

The HJM Modelling Idea for Forward Contracts

This issue together with the rather complicated explicit form of $f(t, \tau)$ are serious ones when we want to fit the model to data, since it becomes a delicate task to derive the distributional properties of the log-returns.

This could, however, be overcome by considering a discretized version of the dynamics giving a representation of the returns instead. Another, simplier way, is to state the explicit dynamics directly rather than the dynamics in differential form.

The HJM Modelling Idea for Forward Contracts

This is the approach suggested by Barndorff-Nielsen (1998) where asset prices are modelled by an exponential NIG Lévy process.

In our context, the analogous modelling perspective would be to define the forward curve dynamics in exponential form under the risk-neutral measure Q as

$$f(t,\tau) = f(0,\tau)\exp\left(\int_0^t a(s,\tau)ds + \int_0^t \sigma(s,\tau)dW(s) + \int_0^t \eta(u,\tau)dJ(u)\right).$$

The HJM Modelling Idea for Forward Contracts

When using the spot market modelling approach, we have the advantage of estimating a market price of risk given by the 'mis-match' netween the spot and forward/swap market. Since both markets are rather liquid, we get a good estimate for the market price of risk. This counts in favor of the spot modelling approach.

However, it is difficult to find good models for the spot dynamics in many markets, and the market price of risk may have a complicated stochastic behavior. Further, the stylised facts of the market price of risk are still not well understood. So far, there are investigations hinting towards a negative market price of risk in the spot end.

The HJM Modelling Idea for Forward Contracts

For instance, Cartea and Figueroa (2005) estimate a negative market price of risk for the England and Wales electricity market, whereas Ollmar (2003) and Weron (2005) fine evidence of a changing sign from negative in the short end to positive in the long end for Nord Pool contracts.

Further, Cartea and Williams (2006) find that the UK gas market may have a price of risk changing sign in the short term and negative for long dated instruments.

HJM Modelling of Forwards

Assume that the forward dynamics under risk-neutral probability Q is

$$\begin{aligned} f(t,\tau) &= f(0,\tau)\exp(\textstyle\int_0^t a(s,\tau)ds + \sum_{k=1}^p \int_0^t \sigma_k(s,\tau)dW_k(s) \\ &+ \textstyle\sum_{j=1}^n \int_0^t \eta_j(u,\tau)dJ_j(u)), \end{aligned}$$

where $a(s,\tau), \sigma_k(s,\tau), \eta_j(u,\tau)$ are real-valued continuous functions on $[0,\tau]\times[0,\mathcal{T}]$, $\mathcal{T}$ is an upper bound for the delivery times in the market. W_k are independent Brownian motions and J_j are independent II processes independent of W_k. The Poisson random measure of J_j is denoted $M_j(dt,dz)$ with the compensator measure $\nu_j(dz,dt)$.

HJM Modelling of Forwards

We find the following risk-neutral dynamics of the forward price, together with a drift condition for $a(u,\tau)$ ensuring the martingale property.

Proposition 6.1. Suppose for each $j = 1, 2, ..., n$, that the exponential integrability condition $\int_0^\tau \int_{|z|\geq 1} \exp(\eta_j(u,\tau)z)\nu_j(dz,du) < \infty$ holds for every $\tau < \mathcal{T}$. Under the drift condition

$$\begin{aligned}\int_0^t [a(s,\tau) &+ \tfrac{1}{2}\textstyle\sum_{k=1}^p \sigma^2(s,\tau)]ds + \sum_{j=1}^n \int_0^t \eta_j(s,\tau)d\gamma_j(s) \\ &+ \textstyle\sum_{j=1}^n \int_0^t \int_R [e^{\eta_j(s,\tau)z} - 1 - \eta_j(s,\tau)z1_{|z|<1}\nu_j(dz,du)] = 0\end{aligned}$$

HJM Modelling of Forwards

The forward price $f(t,\tau)$ has the following dynamics for $\tau < \mathcal{T}$

$$\frac{df(t,\tau)}{f(t-,\tau)} = \sum_{k=1}^{p} \sigma_k(t,\tau)dW_k(t) + \sum_{j=1}^{n} \int_R (e^{\eta_j(t,\tau)z} - 1)\tilde{M}_j(dt,dz),$$

where $\tilde{M}$ is the compensated measure M. Follows from Itô Formula.

The forward price $f(t,\tau)$ is a martingale under the risk-neutral measure.

HJM Modelling of Forwards

We now study the market dynamics of the forward price $f(t,\tau)$ (or under measure P). Let

$$dW_k(t) = \widehat{\theta}_k(t)dt + dB_k(t)$$

and I_j is an II process with drift

$$\gamma_j(t) + \int_0^t \int_{|z|<1} z(e^{\tilde{\theta}_j(u)z} - 1)\nu_j(dz, du),$$

and jump measure $e^{\tilde{\theta}_j(t)}\nu_j(dz, dt)$. Then

$$\begin{aligned} f(t,\tau) &= f(0,\tau)\exp(\textstyle\int_0^t a(s,\tau)ds + \sum_{k=1}^p \int_0^t \sigma_k(s,\tau)\widehat{\theta}_k(s)ds \\ &+ \textstyle\sum_{k=1}^p \int_0^t \sigma_k(s,\tau)dB_k(s) + \sum_{j=1}^n \int_0^t \eta_j(u,\tau)dJ_j(u)), \end{aligned}$$

HJM Modelling of Forwards

Proposition 6.2. The dynamics of $f(t,\tau)$ under P is

$$\begin{aligned}\frac{df(t,\tau)}{f(t-,\tau)} &= \sum_{k=1}^{p} \sigma_k(t,\tau)\widehat{\theta}_k(t)dt + \sum_{k=1}^{p} \sigma_k(t,\tau)dB_k(t) \\ &+ \sum_{j=1}^{n} \int_{|z|<1} (e^{\eta_j(t,\tau)z} - 1)(e^{\tilde{\theta}_j z} - 1)\nu_j(dt,dz) \\ &- \sum_{j=1}^{n} \int_{|z|\geq 1} (e^{\eta_j(t,\tau)z} - 1)\nu_j(dt,dz) \\ &+ \sum_{j=1}^{n} \int_{|z|<1} (e^{\eta_j(t,\tau)z} - 1)\tilde{N}_j(dt,dz) \\ &+ \sum_{j=1}^{n} \int_{|z|\geq 1} (e^{\eta_j(t,\tau)z} - 1)N_j(dt,dz).\end{aligned}$$

Here, N_j is the random measure associated to I_j, and $\tilde{N}_j$ is its compensator. Follows from Itô Formula.

HJM Modelling of Forwards: Two Simple Example

Example 1. $m = n = 1, \quad J = 0$. The drift condition in Prop. 6.1 becomes

$$\int_0^t a(u,\tau) + \frac{1}{2}\sigma^2(u,\tau)du = 0,$$

so,

$$a(u,\tau) = -\frac{1}{2}\sigma^2(u,\tau)du.$$

Hence, the market dynamics of the forward price is

$$\frac{f(t,\tau)}{f(t,\tau)} = \sigma(t,\tau)\widehat{\theta}(t)dt + \sigma(t,\tau)dB(t).$$

HJM Modelling of Forwards: Two Simple Example

Example 2. $m = n = 1$, $d\gamma(u) = \gamma(u)du$ and $\nu(du, dz) = \nu(u, dz)du$.

Then the drift condition of Prop. 6.1 becomes

$$\begin{aligned} a(u,\tau) &+ \tfrac{1}{2}\sigma^2(u,\tau) + \eta(u,\tau)\gamma(u) \\ &+ \textstyle\int_R (e^{\eta(t,\tau)z} - 1 - \eta(t,\tau)z 1_{|z|<1})\nu(t,dz) = 0. \end{aligned}$$

HJM Modelling of Swaps

The electricity and gas markets trade in forward contracts having a delivery period, for which we here will use the common notion swaps. The owner of a swpa contract with delivery over the time interval $[\tau_1, \tau_2]$ would receive a constant flow of the commodity over this period, against a fixed payment per unit. The aim is to derive a price dynamics for such swap contracts based on the HJM approach.

HJM Modelling of Swaps: No-Arbirgae Condition

Consider the swap price $F(t, \tau_1, \tau_N)$ of a contract with delivery over $[\tau_1, \tau_N]$ and N contracts $F(t, \tau_k, \tau_{k+1})$ with delivery over $[\tau_k, \tau_{k+1}]$ for $k = 1, ..., n-1$. Assume that $\tau_1 < \tau_2 < ... < \tau_N$. Then, by appealing to arbitrage arguments, we find the following *no-arbitrage relation* between the swap prices,

$$F(t, \tau_1, \tau_N) = \sum_{k=1}^{N-1} w_k F(t, \tau_k, \tau_{k+1}). \tag{1}$$

Here,

$$w_k = \frac{\int_{\tau_k}^{\tau_{k+1}} w(u)du}{\int_{\tau_1}^{\tau_N} w(u)du}.$$

Any arbitrage-free model of the swap price needs to satisfy the condition (1), at least for those products traded in the market.

HJM Modelling of Swaps

Let $F(t, \tau_1, \tau_2)$ be the price at time t for a swap contract where the underlying is delivered over the period $[\tau_1, \tau_2]$. The swap contract is usually traded over the period of time $t \in [0, \tau_1)$. We introduce the natural extension of the forward dynamics to the case of swap contracts. Suppose that the risk-neutral price dynamics of the swap is

$$\begin{aligned} F(t, \tau_1, \tau_2) &= F(0, \tau_1, \tau_2) \exp(\textstyle\int_0^t A(s, \tau_1, \tau_2) ds \\ &+ \textstyle\sum_{k=1}^p \int_0^t \Sigma_k(s, \tau_1, \tau_2) dW_k(s) \\ &+ \textstyle\sum_{j=1}^n \int_0^t \Upsilon_j(u, \tau_1, \tau_2) dJ_j(u)), \end{aligned}$$

HJM Modelling of Swaps

Proposition 6.3. Suppose for each $j = 1, ..., n$, that the exponential intgrability condition

$$\int_0^{\tau_1} \int_{|z|\geq 1} \exp[\Upsilon_j(u, \tau_1, \tau_2)z]\nu_j(dz, du) < \infty$$

holds for all $0 \leq \tau_1 \leq \tau_2 \leq \mathcal{T}$. Under the drift condition

$$\begin{aligned} \textstyle\int_0^t [A(s,\tau) \quad &+ \quad \tfrac{1}{2}\textstyle\sum_{k=1}^p \Sigma^2(s,\tau)]ds + \sum_{j=1}^n \int_0^t \Upsilon_j(s,\tau)d\gamma_j(s) \\ &+ \quad \textstyle\sum_{j=1}^n \int_0^t \int_R [e^{\Upsilon_j(s,\tau)z} - 1 - \Upsilon_j(s,\tau)z 1_{|z|<1}\nu_j(dz, du)] = 0 \end{aligned}$$

the forward price $F(t, \tau_1, \tau_2)$ has the following dynamics

$$\frac{dF(t,\tau_1,\tau_2)}{F(t-,\tau_1,\tau_2)} = \sum_{k=1}^p \Sigma_k(t,\tau)dW_k(t) + \sum_{j=1}^n \int_R (e^{\Upsilon_j(t,\tau)z} - 1)\tilde{M}_j(dt, dz),$$

HJM Modelling of Swaps: The Continuous No-Arbitrage Relation

The main question we want to answer is whether model for $F(t,\tau_1,\tau_2)$ satisfies the continuous no-arbitrage relation

$$F(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \widehat{w}(u,\tau_1,\tau_2) F(t,u,u)du,$$

where

$$\widehat{w}(u,\tau_1,\tau_2) = \frac{w(u)}{\int_s^t w(v)dv}, \quad w(u) = 1(end \quad period),$$

$$w(u) = \exp(-ru)(during \quad the \quad delivery)$$

We note, $F(t,u,u) = f(t,u)$-forward price. The answer is NO!

HJM Modelling of Swaps: The Continuous No-Arbitrage Relation

Lemma 6.1. Let $n = 0$, and suppose that A, Σ are continuously differentiable in τ_2. Then, if $\partial_k \Sigma_k$ is nonzero for u in a subset of positive measure of $[0, t]$ for at least one k, then the forward price $F(t, \tau_1, \tau_2)$ dynamics does not satisfy the continuous-time no-arbitrage relation

$$F(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \widehat{w}(u, \tau_1, \tau_2) F(t, u, u) du.$$

HJM Modelling of Swaps: The Continuous No-Arbitrage Relation

Proof. $\widehat{w} = 1/(\tau_2 - \tau_1), \quad p = 1$. Differentiating both side of the above relations gives

$$\begin{aligned} F(t,\tau_1,\tau_2)(1/(\tau_2-\tau_1) \quad &- \quad \textstyle\int_0^t \partial_{\tau_2} A du - \int_0^t \partial_{\tau_2} \Sigma dW(u)) \\ &= \quad \tfrac{1}{\tau_2-\tau_1} F(t,\tau_1,\tau_2). \end{aligned}$$

Right-hand side is positive, while the left hand-side may become negative because of the Brownian motion.

HJM Modelling of Swaps: The Continuous No-Arbitrage Relation

Thus, one needs to consider othet models than the exponential class if arbitrage-freeness is to hold in general.

A simple way to obtain such models is to generate them from forward contracts (Sec. 6.3.1, next several slides). An alternative path to the construction of swap models is inspired by the LIBOR models in fixed income theory (Sec. 6.4).

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

In the electricity and gas markets there is no trade with forwards for obvious reasons. However, one may still use the dynamics of such forwards as a building block, with the hope that they will induce reasonable models of the electricity and gas futures price dynamics which are feasible for further analysis.

Suppose that we model a risk-neutral forward price dynamics $f(t,\tau)$ as

$$\begin{aligned} f(t,\tau) &= f(0,\tau)\exp\left(\int_0^t [a(s,\tau) - \tfrac{1}{2}\sigma^2(s,\tau)]ds + \int_0^t \sigma(s,\tau)dW(s)\right) \\ &\times \prod_{s\le t}(1+\eta(s,\tau)\Delta J(s))\exp(-\eta(s,\tau)\Delta J(s)), \end{aligned}$$

where the drift condition for $a(t,u)$ in Prop. 6.1 holds.

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

The swap may be viewed as a continuous flow of forwards, that is

$$F(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \widehat{w}(u,\tau_1,\tau_2) f(t,u) du.$$

We know from Prop. 4.3 that $\lim_{\tau_2 \to \tau_1} F(t,\tau_1,\tau_2) = f(t,\tau_1)$. Thus, the continuous-time no-arbitrage relation also holds. In practice, the swap should be a weighted sum of forward, reflectingthat the smallest delivery period is an hour (or that the spot is really an hourly delivery forward).

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

Let us study the impied price dynamics for the swap contract $F(t, \tau_1, \tau_2)$. To simplify, we take $m = n = 1$. From the dynamics of $f(t, \tau)$ in Prop. 6.1 we have

$$
\begin{aligned}
F(t, \tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} \widehat{w}(u, \tau_1, \tau_2) f(0, u) du \\
&+ \int_{\tau_1}^{\tau_2} \int_0^t \widehat{w}(u, \tau_1, \tau_2) f(s, u) \sigma(s, u) dW(u) du \\
&+ \int_{\tau_1}^{\tau_2} \int_0^t \widehat{w}(u, \tau_1, \tau_2) f(s-, u) \int_R (e^{\eta(s,u)z} - 1) \tilde{M}(dz, ds) du.
\end{aligned}
$$

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

Appealing to the stochastic Fubini Theorem and the classical Fubini-Tonelli Theorem, and using integration by parts formula we find the dynamics of the forward price:

$$\begin{aligned} dF(t,\tau_1,\tau_2) &= \sigma(t,u)F(t,\tau_1,\tau_2)dW(t) \\ &+ F(t-,\tau_1,\tau_2)\int_R (e^{\eta(t,u)z}-1)\tilde{M}(dz,dt) \\ &- \int_{\tau_1}^{\tau_2} \partial_u\sigma(t,u)\frac{\widehat{w}(\tau,\tau_1,\tau_2)}{\widehat{w}(\tau,\tau_1,u)}F(t,\tau_1,u)du dW(t) \\ &- \int_{\tau_1}^{\tau_2}\int_R z\eta(t,u)\partial_u\eta(t,u)\frac{\widehat{w}(\tau,\tau_1,\tau_2)}{\widehat{w}(\tau,\tau_1,u)}F(t-,\tau_1,u)du\tilde{M}(dz,dt). \end{aligned}$$

As we can see the dynamics of the swap is not a GBM as long as the derivatives of σ and η wrt the second variable is nonvanishing.

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

Even though we start out with a GBM for the forward, we end up with a non-Markovian stochastic dynamics for the swap contracts, involving all delivery period contained $[\tau_1, \tau_2]$. Such a dynamics is time consuming to simulate due to the complicated dependency on all swaps with shorter delivery periods, and moreover, it is hard to use it for data estimation or pricing of derivatives.

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

As a degression, we may consider the time dynamics of $F(t, \tau_1, \tau_2)$ as an infinite-dimensional stochastic process. With this interpretation we have in fact a multiplicative structure on the dynamics. However, this requires that we consider the stochastic dynamics of the functional-valued stochastic process $F(F(t, \tau_1, \cdot)$. We refer to DaPrato&Zabczyk (1992) for more details on the theory for infinite-dimensional stochastic processes.

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

Bjerksund, Rasmussen & Stensland (2000) Approximation. They considered electricity market and one-factor model for the forward dynamics without jumps with a specific volatility function given by

$$\sigma(t,\tau) = \frac{\sigma}{\tau - t - b} + c.$$

This volatility function can create a sharp increase in volatility as time to maturity of the contract decreases. It is claimed that the exponentially damping volatility function

$$\sigma(t,\tau) = \sigma \exp(\alpha(\tau - t))$$

implied by a mean reversion dynamics for the spot does not produce an increase in volatility which is sufficiently sharp in the short end of the curve.

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

Bjerksund, Rasmussen & Stensland (2000) Approximation.

They approximated the dynamics for $F(t, \tau_1, \tau_2)$ by

$$\frac{F(t, \tau_1, \tau_2)}{F(t, \tau_1, \tau_2)} = \Sigma(t, \tau_1, \tau_2) dW(t),$$

where Σ is the weighted average volatility of the forward over the delivery period defined as

$$\Sigma(t, \tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \hat{w}(u, \tau_1, \tau_2) \sigma(t, u) du.$$

1) When $w(u) = 1$ and $\sigma(t, \tau) = \sigma \exp(\alpha(\tau - t))$, then the approximated volatility becomes

$$\Sigma(t, \tau_1, \tau_2) = \frac{\sigma}{\alpha(\tau_2 - \tau_1)} [e^{-\alpha(\tau_1 - t)} - e^{-\alpha(\tau_2 - t)}]$$

HJM Modelling of Swaps (Sec. 6.3.1 Swap models based on forwards)

If we consider the volatility specification by Bjerksund, Rasmussen & Stensland (2000), we are led to the expression

$$\Sigma(t, \tau_1, \tau_2) = \frac{\sigma}{(\tau_2 - \tau_1)} \ln\left(\frac{\tau_2 - t + b}{\tau_1 - t + b}\right) + c.$$

We will return to this model in Chapter 9 when considering option pricing and the Black-76 formula.

The Market Models

The LIBOR (London Inter Bank Offer Rate) models in interest rate theory form a flexible class of dynamical models for LIBOR rates matching the implied volatility of captions or swaptions traded in the market.

We propose a similar modelling approach for the swap price dynamics in the energy markat, where the idea is to construct a dynamics for the *traded* contracts matching with the observed volatility term structure. Since the option markets on electricity and gas are rather thin, we want to estimate the model on the swap prices themselves.

The Market Models

The difference from the HJM approach discussed above is that we consider models only for the products traded in the market, and thereby make the possible range of models much wider since we avoid the continuous-time no-arbirage condition. We refer to this as the *market model.*

When intorducing the market models, we first single out the contracts which can not be decomposed into contracts with smaller delivery periods. For example, we single out the quarterly contracts, and do not model the yearly contract directly, but rather as a sum of the quarterly ones. These contracts will be called the *basic contracts*.

The Market Models

Let $\{[\tau_1^b, \tau_1^e], ..., [\tau_1^C, \tau_1^C]\}$ be a sequence of delivery periods for the different basic contracts, for $c = 1, 2, ..., C$. Also, $\tau_c^e \leq \tau_{c+1}^b$. A typical example being monthly contarcts, that is, contracts with delivery each month over the year. In this case, assuming that we are at the beginning of the year, $\tau_c^b = (c-1)/12$ and $\tau_c^e = c/12$, with $c = 1, 2, ..., C$ and with time measured in years. We state the forward dynamics for each contract under risk neutral probability, in line with the HJM modelling approach discussed in the sections above.

The Market Models

Denote by $F_c(t) := F(t, \tau_c^b, \tau_c^e)$ for $c = 1, ..., C$, and assume that the risk-neutral explicit dynamics is

$$\begin{array}{rcl} F_c(t & = & F_c(0)\exp(\int_0^t A_c(s)ds \\ & + & \sum_{k=1}^p \int_0^t \Sigma_{c,k}(sdW_k(s) \\ & + & \sum_{j=1}^n \int_0^t \Upsilon_{c,j}(s)dJ_j(s)). \end{array}$$

The functions $A_c, \Sigma_{c,k}, \Upsilon_{c,j}$ are continuous real-valued on $[0, \tau_c^b]$, since we assume that trading of the contracts ends at the beginning of the delivery period.

The Market Models

Now, we state the drift condition that ensures the swap price dynamics to be a amrtingale.

Proposition 6.4. Assume that the exponential integrability condition for each $j = 1, 2..., n$

$$\int_0^{\tau_c^b} \int_{|z|\geq 1} \exp(\Upsilon_{c,j}(t)z)\nu_c(dz, dt) < +\infty$$

holds. Under the drift condition

$$\begin{aligned} \int_0^t A_c(u)du &+ \tfrac{1}{2}\sum_{k=1}^p \Sigma_{c,k}^2(u)du + \sum_{j=1}^n \int_0^t \Upsilon_{c,j}(u)d\gamma_j(u) \\ &+ \sum_{j=1}^n \int_0^t \int_R (e^{\Upsilon_{c,j}(u)} - 1 - \Upsilon_{c,j}(u)z1_{|z|<1})\nu_j(dz, du) = 0 \end{aligned}$$

for every $t \leq \tau_c^b$, the swap price $F_c(t)$ has for $t \leq \tau_c^b$ the following dynamics

The Market Models

$$\frac{dF_c(t)}{F_c(t)} = \sum_{k=1}^{p} \Sigma_{c,k}(t) dW_k(t) + \sum_{j=1}^{n} \int_R (e^{\Upsilon_{c,j}(t) z} - 1) \tilde{M}_j(dt, dz).$$

The Market Models: Examples

Example 1: Benth & Koekebakker (2005). A simple one-factor model without jumps:

$$\frac{dF_c(t)}{F_c(t)} = \sigma_c(t)dW(t),$$

where c labels the different contracts, and the volatility $\sigma_c(t)$ is a function explicitly dependent on the delivery period of the contract in question, $[\tau_c^b, \tau_c^e]$:

$$\sigma_c(t) = \frac{1}{\tau_c^e - \tau_c^b} \int_{\tau_c^b}^{\tau_c^e} \sigma(t,u)du + s(t),$$

where $s(t)$ is a seasonality function defined as a truncated Fourier series and $\sigma(t,u) = \sigma \exp(-\alpha(u-t))$.

The Market Models: Examples

To fit this volatility structure to observed data, we need to have the P dynamics available. In Benth & Koekebakker (2005) it was assumed a constant market price of risk, that is, a constant θ, leading to the P dynamics

$$\frac{dF_c(t)}{F_c(t)} = \theta\sigma_c(t)dt + \sigma_c(t)dB(t).$$

The Market Models: Examples

Example 1: Benth & Koekebakker (2005). Such a one-factor model is rather simplistic, and unlikely to capture all stylized facts of the electricity futures price dynamics.

In view of the findings in Frestad (2007), contracts with different lenghts of delivery and delivery at different times of the year are not perfectly correlated, but show a rather complicated pattern of dependency. This calls for multi-factor models. Kjaer & Ronn (2006) use a forward model to study gas futures returns on NYMEX, where a non-stationary correlatiion structure is observed.

The Market Models: Examples

Example 2: Kiesel, Schindlmayer & Borger (2006). A two-facotr model for electricity futures prices at EEX. They model the basic contracts as

$$\frac{dF_c(t)}{F_c(t)} = \sigma_{c,1}(t)dW_1(t) + \sigma_{c,2}(t)dW_2(t),$$

where c lables the different contacts, and

$$\sigma_{c,1}(t) = sigma_1 \,\mathsf{exp}(-\alpha(\tau_c^b - t)), \quad \sigma_{c,2}(t) = \sigma_2.$$

The Market Models: Examples

Example 2: Kiesel, Schindlmayer & Borger (2006). $\sigma_{c,1}(t)$ mimics the volatility term structure arising from a mean reversion model, while the second volatility models the non-stationary part. The volatility in the electricity futures contract will decay exponentially towards $\sigma_{c,2}$ with increasing time to maturity of the contract. They estimate the model on the implied volatility term structure of monthly, quarterly and yearly contracts.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

Empirical results in Benth and Koekebakker (2005), Frestad, Benth and Koekebakker (2007) and Green (2006) suggest that the logreturns of electricity futures prices are far from being normally distributed. This exludes the GBM models introduced above. In fact, the logreturns are heavy-tailed, rather symmetric, and peaked in the center of the distribution. The analysis in Frestad, Benth and Koekebakker (2007) and Green (2006) points towards the use of NIG models for the logreturns.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

Consider one swap contract specified by a given c. We defined our market models directly under the risk-neutral probability Q. If we want to have a specified distribution under the market probability P, we need to translate the dynamics by using the Esscher transform. Thus, suppose for the given contract that $p = 0, n = 1, \Upsilon = 1$. Further, we assume that $A_c = 0$ and let J by a Lévy process such that $J(1)$ is equivalent to knowing the cumulant function ψ.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

The dynamics under Q of the forward price $F_c(t)$ is

$$F_c(t) = F_c(0)\,\exp(J(t)),$$

where $J(t)$ must satusfy a martingale condition. The cumulant function ψ associated with the Lévy process J has to be so that $e^{t\psi(-i)} = 1$, or, $\tilde{\psi}(-i) = 0$. Introducing a market price of risk $\tilde{\theta}(t)$, we get the complete characteristics for J under P from the characteristics under Q.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

Returning to our prime example, the NIG distribution, we find that under a constant Esscher transform $\tilde{\theta}$ of J, the assumption that $J(1) \equiv NIG(\alpha, \beta, \delta, \mu)$ under Q becomes $J(1) \equiv NIG(\alpha, \beta + \tilde{\theta}, \delta, \mu)$ under P. Recalling the explicit cumulant function for the NIG distribution the cumulant condition $\tilde{\psi}(-i) = 0$ thus becomes

$$\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 + (\beta + 1)^2}) = 0.$$

When performing a ML estimation, all the parameters $\alpha, \beta + \tilde{\theta}, \delta, \mu$ will be fitted to the data, leaving β and $\tilde{\theta}$ unestimated. However, the last condition yields a condition on β in terms of μ, δ, α. Thus, we can find the market price of risk by solving it for β.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

We have several possible models which can be employed to model contracts $F_c, c = 1, ..., C$. We use a one-factor model in the sense that we state the dynamics

$$F_c(t) = F_c(0)\exp(\Upsilon_c J(t))$$

for constants Υ_c. In this case, the martingale condition becomes $\psi(-i\Upsilon_c) = 0$, for all $c = 1, ..., C$. This means, in particular for the NIG case, that

$$\mu\Upsilon_c + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + \Upsilon_c)^2}) = 0.$$

The problem is that Υ_c is varying with the contract. Letting Υ_c be time-dependent only makes the situation wrse.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

To solve this, we introduce several independent Lévy processes J_j by stating a multi-factor model. Each contract depends on one or more of the factors in the following fashion. For $c = 1, ..., C$,

$$F_c(t) = F_c(0)\,\exp(\sum_{j=1}^{n} \Upsilon_{c,j} J_j(t))$$

with constants $\Upsilon_{c,j}$. The problem here is: if would like to use NIG for J_j then a sum of independent NIG distributed r.v. is NOT NIG distributed. Hence, we lose the attractive property of the explicit knowledge of the marginal distribution and can not be sure that the theoretical model has the desirable distribution. We can perform a numerical fitting, but this may become a rather complicated task.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

An alternative is to assume that $\mathbf{J}(1) = (J_1(1), ..., J_C(1))$ is distributed according to a multiplicative NIG distribution. The multivariate NIG distribution is defined via its cumulant function, which means that $\mathbf{J}(1)$ is $NIG_C(\alpha, \beta, \delta, \mu, \Delta)$ if it has a cumulant function being equal to

$$\begin{aligned} \psi_{mNIG}(\theta) &= \ln E[e^{i\theta \mathbf{J}(1)}] = -i\mu\theta' \\ &+ \delta(\sqrt{\alpha^2 - \beta\Delta\beta'} - \sqrt{\alpha^2 - (\beta + i\theta)\Delta(\beta + i\theta)'}). \end{aligned}$$

Here, $\mu, \beta \in R^C, \delta > 0, \alpha > 0$, and $\theta \in R^C$. Finally, Δ is a positive definite matrix in $R^{C \times C}$ with determinant equal to one, and u' means the transpose of u. We refer to Rydberg (1997) for a discussion of properties of the multivariate NIG distribution.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

The model becomes

$$F_c(t) = F_c(0)\exp(J_c(t)),$$

for $c = 1, ..., C$. We chose as many NIG as contracts. From the cumulant function above we get

$$\mathbf{e}_c\mu + \delta(\sqrt{\alpha^2 - \beta\Delta\beta'} - \sqrt{\alpha^2 - (\beta + \mathbf{e}_c)\Delta(\beta + i\mathbf{e}_c)'}) = 0,$$

where $\mathbf{e}_c$ is the cth unit vector in R^C.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

Let the density process of the Radon-Nikodym derivative of P wrt Q^θ be (for constant $\tilde{\theta} \in R^C$)

$$\frac{dP}{DQ}|_{\mathcal{F}_t} = \exp(\tilde{\theta}\mathbf{J}'(t) - \psi_{mNIG}(-i\tilde{\theta})t).$$

It is easy to see that in this case the distribution of $\mathbf{J}(1)$ is a multivariate NIG, with the same parameters except for the skewness β, which under P becomes $\beta + \tilde{\theta}$, analogous to the univariate case.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

The multivariate GH distribution, and the particular case of NIG, was proposed and empirically analysed by Borger et al. (2007) as a joint model for electricity, gas and CO_2 returns. The drawback with the multivariate NIG is the high dimensionality which may cause numerical problems when fitting the likelihood function to a set of logreturn data.

The Market Models: Modelling with Jump Processes (Sec. 6.4.1)

Following recent theory of copulas, we may keep to a much simplier path. First, we fit each contarct marginally with the desired Lévy model. The next step is to introduce a dependency structure by using a so-called Lévy copula presented in Kallsen and Tankov (2006). The Lévy copula creates a C-dimensional Lévy process $(J_1(t), ..., J_C(t))$ from C Lévy processes J_j with marginal distribution. This way of modelling a multivariate Lévy process is very flexible, since we can do it first marginally, and then model the dependency.

In Chapter 8, we come back to more detailed discussion on the statistical properties of electricity futures prices.

Conclusion

1. Intro
2. The HJM Modelling Idea for Forward Contracts
3. HJM Modelling of Forwards
4. HJM Modelling of Swaps (Swap models based on forwards)
5. The Market Models (Modelling with jump processes)

The End of Chapter 6!

Thank You for Your Time and Attention!