Chapter 10: Analysis of Temperature Derivatives *

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*Book Review: 'Stochastic Modeling of Electricity and Related Markets' by F. Benth, J. Benth & S. Koekebakker, 2008, World Sci. Publ.

Outline of Presentation

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3. Modelling the Dynamics of Temperature (sec. 10.2)

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Intro: Abstract

The final Chapter 10 is devoted to the market for temperature futures. They present continuous-time mean reversion models being generalizations of autoregressive moving average time series. Applying these to temperature data, they find that the 'volatility' of temperature has a clear seasonal pattern. The temperature models allow for rather explicit pricing of the typical futures traded on Chicago ME. The chapter includes a thorough empirical analysis of Stockholm temperature data in view of the proposed models.

Some Preliminaries on Temperature Futures (sec. 10.1)

In what follows, they derive expression for the dynamics of futures prices based on a mean-reverting AR model for the temperature evolution.

The CDD-*cooling-degree day*-(and analogously the HDD-*heating-degree days*) over a measurement period $[\tau_1, \tau_2]$ is defined as

$$\int_{\tau_1}^{\tau_2} \max(T(s) - c, 0) ds,$$

where $T(t)$ is the instantaneous temperature at time t.

Some Preliminaries on Temperature Futures (sec. 10.1)

The CAT-*cumulative average temperature*-and PRIM-*Pasific Rim*-indices over the same period are

$$\int_{\tau_1}^{\tau_2} T(s)ds$$

and

$$\frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2} T(s)ds,$$

respectively.

Some Preliminaries on Temperature Futures (sec. 10.1)

The buyer of a CDD futures contract will receive the amount $\int_{\tau_1}^{\tau_2} \max(T(s) - c, 0)ds$ at the end of the measurement period $[\tau_1, \tau_2]$. In return, the buyer pays the CDD futures price $F_{CDD}(t, \tau_1, \tau_2)$ if the contract was entered at time $t \leq \tau_1$. The profit from this trade is therefore

$$\int_{\tau_1}^{\tau_2} \max(T(s) - c, 0)ds - F_{CDD}(t, \tau_1, \tau_2).$$

From arbitrage theory, the CDD futures price is given by the eqaution

$$0 = e^{-r(\tau_2 - t)} E_Q[\int_{\tau_1}^{\tau_2} \max(T(s) - c, 0)ds - F_{CDD}(t, \tau_1, \tau_2)|\mathcal{F}_t],$$

with a constant risk-free rate of return r and a risk-neutral probability Q.

Some Preliminaries on Temperature Futures (sec. 10.1)

Since temperature (and therefore the CDD index) is not tradeable, any probability Q being equivalent to the objective probability P is a risk-neutral probability. Later, they shall specify a subclass of such probabilities via the Girsanov transform. The CDD futures price is adapted, and thus we derive it as the conditional risk-neutral expected payment from the CDD index

$$F_{CDD}(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} \max(T(s)-c,0)ds|\mathcal{F}_t].$$

Some Preliminaries on Temperature Futures (sec. 10.1)
Analogously, we find that

$$F_{HDD}(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} \max(T(s)-c)ds|\mathcal{F}_t].$$

Similar derivations lead to the CAD and PRIM futures prices being

$$F_{CAT}(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} T(s)ds|\mathcal{F}_t].$$

and

$$F_{CAT}(t,\tau_1,\tau_2) = E_Q[\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} T(s)ds|\mathcal{F}_t].$$

Some Preliminaries on Temperature Futures (sec. 10.1)

We have the following useful CDD-HDD parity.

Proposition 10.1 The CDD and HDD futures prices are linked by the relation

$$F_{HDD}(t,\tau_1,\tau_2) = c(\tau_1 - \tau_2) - F_{CAT}(t,\tau_1,\tau_2) + F_{CDD}(t,\tau_1,\tau_2)$$

Proof. The result follows from

$$\max(c - x, 0) = c - x + \max(x - c, 0).$$

Some Preliminaries on Temperature Futures (sec. 10.1)

In this Chapter they focus on deriving the CDD and CAT futures prices. The Proposition 10.1 above readily gives the HDD futures price as long as we know the CDD and CAT futures prices.

We recall that a Frost Day at Amsterdam airport Schiphol is defined as observed frost in the morning. More specifically, we have (with time measured in days) that

$$\begin{aligned} FD(t) &= 1\{[(T(t+7/24) \leq -3.5) \cup (T(t+10/24) \leq -1.5)] \\ &\cup \ [(T(t+7/24) \leq -0.5) \cap (T(t+10/24) \leq -0.5)]\}. \end{aligned}$$

Some Preliminaries on Temperature Futures (sec. 10.1)

The Frost Day index over a measurement period $[\tau_1, \tau_2]$ is

$$\sum_{t=\tau_1}^{\tau_2} FD(t).$$

Using the same procedure as above, we may derive the Frost Day index futures price as

$$F_{FDI}(t, \tau_1, \tau_2) = E_Q[\sum_{s=\tau_1}^{\tau_2} FD(s)|\mathcal{F}_t].$$

Some Preliminaries on Temperature Futures (sec. 10.1)

There exist several methodologies to assess derivatives prices on different temperature indices.

They refer to Geman (1999), Geman and Leonardi (2005), Jewson and Brix (2005) for detailed accounts on some established methods (including theirs).

Davis (2001) proposed an approach based on marginal utility to price options on CDDs and HDDs, whereas Platen and West (2005) suggest an equilibrium method based on a world index for temperature derivatives valuation.

Modelling the Dynamics of Temperature (sec. 10.2)

They present here a class of stochastic processes generalizing the multi-factor OU models which were presented in Chapter 3. The class of models is called continuus AR (CAR) processes, since they are AR stochastic processes in continuous time. The CAR model is a subclass of the more general CARMA (continuous autoregressive moving-average) models introduced and studied by Brockwell and Marquardt (2005). Such models are particularly suitable to capture the evolution of temperature through time. They extend the models to allow for seasonality in the residual variance.

Modelling the Dynamics of Temperature (sec. 10.2): The CAR(p) Model with Seasonality (sec. 10.2.1)

Let $\mathbf{X}(t)$ be a stochastic process in R^p for $p \geq 1$ defined by the vectorial OU stochastic process

$$d\mathbf{X}(t) = A\mathbf{X}(t)dt + \mathbf{e}_p(t)\sigma(t)dB(t),$$

where $\mathbf{e}_p(t)$ is the pth unit vector in R^p, $\sigma(t)$ is the volatility of the temperature dynamics.

Modelling the Dynamics of Temperature (sec. 10.2): The CAR(p) Model with Seasonality (sec. 10.2.1)

Further, we denote by A the $p \times p$ matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \dots & -\alpha_1, \end{bmatrix}$$

where $\alpha_k, k = 1, ..., p$ are positive constants.

Modelling the Dynamics of Temperature (sec. 10.2)

They assume further that the seasonal function $\Lambda(t) : [0, \mathcal{T}] \rightarrow R$ is bounded and continuously differentiable. We introduce the follwoing CAR(p) model for the temperature dynamics

$$T(t) = \Lambda(t) + X_1(t),$$

where X_q is the qth coordinate of the vector $\mathbf{X}$.

Modelling the Dynamics of Temperature (sec. 10.2)

Lemma 10.1 The stochastic process $\mathbf{X}(t)$ has the explicit form

$$\mathbf{X}(s) = \exp(A(s-t))\mathbf{x} + \int_t^s \exp(A(s-u))\mathbf{e}_p \sigma(u) dB(u),$$

for $s \geq t \geq 0$ and $\mathbf{X}(t) = \mathbf{x} \in R^p$.

Modelling the Dynamics of Temperature (sec. 10.2)

Using Girsanov Theorem with

$$B^{\theta}(t) = B(t) - \int_0^t \theta(u)du,$$

the dynamics of $\mathbf{X}(t)$ under Q^{θ} becomes

$$d\mathbf{X}(t) = (A\mathbf{X}(t) + \mathbf{e}_p\sigma(t)\theta(t))dt + \mathbf{e}_p(t)\sigma(t)dB^{\theta}(t).$$

Modelling the Dynamics of Temperature (sec. 10.2)

The application of the multidimensional Ito Formula gives the following dynamics for $\mathbf{X}(t)$ under Q^θ :

$$\begin{aligned}\mathbf{X}(s) &= \exp(A(s-t))\mathbf{x} + \int_t^s \exp(A(s-u))\mathbf{e}_p\sigma(u)\theta(u)du \\ &+ \int_t^s \exp(A(s-u))\mathbf{e}_p\sigma(u)dB^\theta(u).\end{aligned}$$

In the empirical studies of temperature data they use the dynamics of $T(t)$ under the market measure P as the model, while the risk-neutral version of the above formula is the appropriate model when analysing futures prices and options on these.

Modelling the Dynamics of Temperature (sec. 10.2): A Link to Time Series (10.2.2)

Consider the special case $p = 1$, where the matrix A simply becomes the constant $-\alpha_1$. The dynamics of $\mathbf{X}(t) = X_1(t)$ is then

$$dX_1(t) = -\alpha_1 X_1(t) + \sigma(t) dB(t).$$

It is known that this process in discrete-time corresponds to an AR(1) process.

Modelling the Dynamics of Temperature (sec. 10.2): A Link to Time Series (10.2.2)

Consider now a general CAR(p) process $\mathbf{X}(t)$ and AR(p) process. First, we have for $q = 1, 2, ..., p-1$, that

$$dX_q(t) = X_{q+1}(t)dt$$

and

$$dX_p(t) = -\sum_{q=1}^{p} \alpha_{p-q+1} X_q(t)dt + \sigma(t)dB(t).$$

Modelling the Dynamics of Temperature (sec. 10.2): A Link to Time Series (10.2.2)

An Euler approximation of the SDE with time step one, leads to a time series $x_p(t), t = 0, 1, ...$ of the following form

$$x_p(t+1) - x_p(t) = -\sum_{q=1}^{p} \alpha_{p-q+1} X_q(t)dt + \sigma(t)\epsilon(t)$$

and

$$x_q(t+r) - x_q(t+r-1) = x_{q+1}(t+r-1)$$

for $q = 1, 2, ..., p-1$ and $r \geq 1$.

Modelling the Dynamics of Temperature (sec. 10.2): A Link to Time Series (10.2.2)

Iterating this, we get the following

Lemma 10.2. For $q = 1, ..., p-1$ it holds

$$x_{q+1}(t) = \sum_{k=0}^{q} (-1)^k b_k^q x_1(t+q-k).$$

Here, the coefficients b_k^q are defined recursively as

$$b_k^q = b_{k-1}^{q-1} + b_k^{q-1}, \quad k = 1, ..., p-1, \quad q \geq 2,$$

and $b_0^q = b_q^q = 1$ for $q = 0, 1, ..., p$. Further, we have that

$$x_p(t+1) - x_p(t) = \sum_{k=0}^{p} (-1)^k b_k^p x_1(t+p-k).$$

Modelling the Dynamics of Temperature (sec. 10.2): A Link to Time Series (10.2.2)

Inserting the expression for x_q in terms of x_1 derived in the Lemma above, we have the following recursive expression for x_1

$$\begin{aligned}\sum_{k=0}^{p}(& - & 1)^k b_k^q x_1(t+p-k) = \\ & - & \sum_{q=1}^{p} \alpha_{p-q+1} \sum_{k=0}^{q-1}(-1)^k b_k^{q-1} x_1(t+q-1-k) + \sigma(t)\epsilon(t).\end{aligned}$$

Observe that the expression includes all the terms $x_1(t+p), x_1(t+p-1), ..., x_1(t)$ in a linear fashion, and thus defines an AR(p) process.

Modelling the Dynamics of Temperature (sec. 10.2): A Link to Time Series (10.2.2)

Example 10.1. AR(2) model. Let $p = 2$. Then we have

$$x_1(t+2) = (2-\alpha_1)x_1(t+1) + (\alpha_1 - \alpha_2 - 1)x_1(t) + \sigma(t)\epsilon(t).$$

Example 10.1. AR(3) model. Let $p = 3$. Then we have

$$\begin{aligned} x_1(t+3) &= (3-\alpha_1)x_1(t+2) + (2\alpha_1 - \alpha_2 - 3)x_1(t+1) \\ &+ (\alpha_2 + 1 - \alpha_1 - \alpha_3)x_1(t) + \sigma(t)\epsilon(t). \end{aligned}$$

The utilization the explicit connection between AR(3) and CAR(3) models is used when analysing temperature derivatives for Stockholm.

Conclusion

1. Intro: Abstract

2. Some Preliminaries on Temperature Futures (sec. 10.1)

3. Modelling the Dynamics of Temperature (sec. 10.2)

Thank You for Your Time and Attention!