Chapter 4: Pricing of Forwards and Swaps Based on Spot Price *

Anatoliy Swishchuk
University of Calgary

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*Book Review: 'Stochastic Modeling of Electricity and Related Markets' by F. Benth, J. Benth & S. Koekebakker, 2008, World Sci. Publ.

Outline of Presentation

1. Risk-Neutral Forward and Swaps Price Modelling: Risk-Neutral Probabilities and Esscher Transform, Examples

2. Currency conversion for forward and swap prices

3. Pricing of Forwards: The Geometric and Arithmetic Cases

4. Pricing of Swaps: The Geometric and Arithmetic Cases

Risk-Neutral Forward and Swap Price Modelling

Suppose we buy a forward contract at time t promising future delivery of some underlying spot product with price dynamics $S(t)$:

$$S(t) = \Lambda(t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t) \quad (arithmetic \quad case)$$

or

$$S(t) = \Lambda(t) \exp(\sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t)) \quad (geometric \quad case)$$

Risk-Neutral Forward and Swap Price Modelling

Here:

$$dX_i(t) = (\mu_i(t) - \alpha_i(t) X_i(t)) X_i(t) dt + \sum_{k=1}^{p} \sigma_{ik} dB_k(t), \quad i = 1, 2, ..., m,$$

and

$$dY_j(t) = (\delta_j(t) - \beta_j(t)) Y_j(t) dt + \eta_j(t) dI_j(t), \quad j = 1, 2, ..., n.$$

Risk-Neutral Forward and Swap Price Modelling: Forward Pricing

When entering the forward contract, one agrees on a future deliverty time and the price to be paid for receiving the underlying. Suppose that the delivery time is τ, with $0 \leq t \leq \tau < +\infty$, and that the agreed price to pay upon delivery is $f(t, \tau)$. At time τ, we will effectively receive a (possibly negative) payment

$$S(\tau) - f(t, \tau).$$

It is costless to enter such contracts, which gives us a relation where we can extract the forward price:

$$e^{-r(\tau - t)} E_Q[S(\tau) - f(t, \tau) | \mathcal{F}_t] = 0.$$

Risk-Neutral Forward and Swap Price Modelling: Forward Pricing

$$f(t,\tau) = E_Q[S(\tau)|\mathcal{F}_t] -$$

fundamental pricing relation between the spot and forward price. Since the energy markets are incomplete, the choice of martingale measure Q is open.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

Let us consider swaps, using the electricity market as the typical example. The buer of an electricity futures receives power during a setlement period (physically or financially), against paying a fixed price per MWh. The time t value of the payoff from the continuous flow electricity is given as

$$\int_{\tau_1}^{\tau_2} e^{-r(u-t)}(S(u) - F(t, \tau_1, \tau_2))du,$$

where $F(t, \tau_1, \tau_2)$ is the electricity futures price at time t for the delivery period $[\tau_1, \tau_2]$ with $\tau_1 \leq \tau_2$.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

Since it is costless to enter an electricity futures contract, the risk-neutral price is defined by the equation

$$e^{-rt}E_Q[\int_{\tau_1}^{\tau_2} e^{-r(u-t)}(S(u) - F(t,\tau_1,\tau_2))du|\mathcal{F}_t] = 0.$$

As long as F is adapted we have

$$F(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} \frac{re^{-ru}}{e^{-r\tau_1} - e^{-r\tau_2}} S(u)du|\mathcal{F}_t].$$

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

One may have that the settlement takes place finacially at the end of the delivery period τ_2. The payoff from the contract at time τ_2 is then

$$e^{-r\tau_2} E_Q[\int_{\tau_1}^{\tau_2} (S(u) - F(t,\tau_1,\tau_2))du|\mathcal{F}_t] = 0,$$

which yields an electricity futures price

$$F(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} \frac{1}{\tau_2 - \tau_1} S(u)du|\mathcal{F}_t].$$

The same considerations could be done for gas futures contracts, and in the following we refer to $F(t,\tau_1,\tau_2)$ simply as the *swap price*.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

Let us introduce a weight function $\widehat{w}(u)$, being equal to one if the swap is settled at the end of the delivery period, or $\widehat{w}(u) = \exp(-ru)$ if the contract is settled continuously over the delivery period. Define the function

$$w(u,s,t) = \frac{\widehat{w}(u)}{\int_s^t \widehat{w}(v)dv},$$

where $0 \leq u \leq s \leq t$. Observe that $w = 1/(t-s)$, when $\widehat{w} = 1$, while we have

$$w(u,s,t) = \frac{re^{-ru}}{e^{-rs} - e^{-rt}},$$

for the case when $\widehat{w} = \exp(-ru)$. We note that $\int_s^t w du = 1$.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

In general, we can write the link between a swap contract and the underlying spot as

$$F(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2)S(u)du|\mathcal{F}_t].$$

Proposition 4.1. Suppose $E_Q[\int_{\tau_1}^{\tau_2} |w(u,\tau_1,\tau_2)S(u)|du] < +\infty$. It holds that

$$F(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2)f(t,u)du.$$

This means that holding a swap contract can be considered as holding a (weighted) continuous stream of forwards.

Risk-Neutral Forward and Swap Price Modelling: Forward Pricing

It is known that the forward price at delivery coincides with the spot price of the underlying. We have a convergence of forward prices to the spot price when time approaches delivery. We recall the following result for contracts of forward type.

Lemma 4.1. Suppose that $E_Q[|S(\tau)|] < +\infty$. It holds that

$$\lim_{t \to \tau} f(t, \tau) = S(\tau).$$

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

In the electricity and gas markets, say, where delivery takes place over a period of time rather than at a fixed point, we do not observe a convergence of swap prices to the spot at delivery. The reason for this is easily seen from the connection between forwards and swaps stated in Prop. 4.1.

Proposition 4.2. Suppose $E_Q[\int_{\tau_1}^{\tau_2} |w(u,\tau_1,\tau_2)S(u)|du] < +\infty$. The it holds that, a.s.,

$$\lim_{t\to\tau_1} F(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2)f(\tau_1,u)du,$$

which is different from $S(\tau_1)$ a.s., as long as $S(t)$ is not a Q martingale.

Risk-Neutral Forward and Swap Price Modelling: Swap Pricing

The next Proposition confirms that a swap contract delivering the commodity over a time period which collapses into a single point conicides with a forward.

Proposition 4.3. Suppose $E_Q[\int_{\tau_1}^{\tau_2} |w(u,\tau_1,\tau_2)S(u)|du] < +\infty$. The it holds that

$$\lim_{\tau_2 \to \tau_1} F(t,\tau_1,\tau_2) = f(t,\tau_1).$$

These relationship between the spot, forwards and swaps were first discussed in the contaxt of electricity markets by Vehlilainen (2002).

Risk-Neutral Probabilities and the Esscher Transform

The Esscher transform is a generalization of the Girsanov transform of Brownian motion to jump processes.

The Esscher transform is preserving the distributional properties of the jump process in the sense of transforming the cumulant function by a linear chanage of the argument.

Effectively, the Esscher transform yields an explicit change of measure, where we have access to the characteristics of the jump processes I_j also under the new risk-neutral measure.

Risk-Neutral Probabilities and the Esscher Transform

It was intorduced by Esscher (1932) to study risk theory and used by Gerber and Siu (1994) for derivatives pricing. Suppose we have f probability density and θ is a real number. Then, as long as $\int_R e^{\theta y} f(y)dy < +\infty$, we can define a new probability density

$$f(x;\theta) = \frac{e^{\theta x} f(x)}{\int_R e^{\theta y} f(y)dy}.$$

We are going to generalize this approach to II processes including time-dependent parameters $\theta(t)$.

Risk-Neutral Probabilities and the Esscher Transform

Let now $\theta(t)$ be a $(p+n)$-dimensional vector of real-valued continuous functions on $[0, T]$

$$\theta(t) = (\widehat{\theta}_1(t), ..., \widehat{\theta}_p(t), \tilde{\theta}_1(t), ..., \tilde{\theta}_n(t)).$$

Define for $0 \leq t \leq \tau$ the stochastic exponential

$$Z^{\theta}(t) = \prod_{k=1}^{p} \widehat{Z}_k^{\theta}(t) \times \prod_{j=1}^{n} \tilde{Z}_j^{\theta}(t),$$

where

Risk-Neutral Probabilities and the Esscher Transform

$$\widehat{Z}_k^\theta(t) = \exp(\int_0^t \widehat{\theta}_k(s) dB_k(s) - \frac{1}{2}\int_0^t \widehat{\theta}_k^2(s) ds), \quad k = 1, 2, ..., p,$$

and

$$\tilde{Z}_j^\theta(t) = \exp(\int_0^t \tilde{\theta}_j(s) dI_j(s) - \phi_j(0, t; \tilde{\theta}_j(\cdot))), \quad j = 1, 2, ..., n.$$

We note, that $\widehat{Z}_k^\theta(t)$ and $\tilde{Z}_j^\theta(t)$ are a positive local martingales with expectation equals to one. Hence, we can define an equivalent probability measure Q^θ such that Z^{theta} is the density process of the Radon-Nikodym derivative dQ^θ/dP, that is,

$$\frac{dQ^\theta}{dP}|_{\mathcal{F}_t} = Z^\theta(t).$$

Risk-Neutral Probabilities and the Esscher Transform

The expectation operator wrt the probability Q^θ is denoted by $E_\theta[\cdot]$. We observe that the Radon-Nikodym derivative dQ^θ/dP can be factorized as

$$\frac{dQ^\theta}{dP}|_{\mathcal{F}_t} = \prod_{k=1}^{p} \hat{Z}_k^\theta(t) \times \prod_{j=1}^{n} \tilde{Z}_j^\theta(t).$$

Hence, we associate a price of risk to each random source given by the Brownian motion B_k and the jump factors I_j, $k = 1, 2, ..., p, j = 1, 2, ..., n$ in the model of spot price.

Risk-Neutral Probabilities and the Esscher Transform

Let us study how the characteristics of B and I are changing when we apply the Esscher transform.

Proposition 4.4. Under measure Q^θ the processes

$$B_k^\theta(t) = B_k(t) - \int_0^t \widehat{\theta}_k(s)ds$$

are Brownian motions, $k = 1, 2, ..., p$. Furthermore, for each $j = 1, 2, ..., n$, $I_j(t)$ ia an II process with drift

$$\gamma_j(t) + \int_0^t \int_{|z|<1} z(e^{\tilde{\theta}_j(u)z} - 1)l_j(dz, du),$$

and compensatior measure $e^{\tilde{\theta}_j(t)z}l_j(dz, dt)$. We denote the new random jump measure by N_j^θ and its compensator by $\tilde{N}_j^\theta$.

Risk-Neutral Probabilities and the Esscher Transform

We note, that $N_j^\theta = N_j$. However, $\tilde{N}_j^\theta$ is not coinciding with $\tilde{N}_j$:

$$\begin{aligned}\tilde{N}_j^\theta &= N_j^\theta - e^{\tilde{\theta}_j(t)z} l_j(dz,dt)\\ &= N_j - l_j - (e^{\tilde{\theta}_j(t)z} - 1)l_j\\ &= \tilde{N} - (e^{\tilde{\theta}_j(t)z} - 1)l_j.\end{aligned}$$

Hence, $\tilde{N}_j$ translates to $\tilde{N}_j^\theta$ by sybtraction of a drift, exactly as the Girsanov transform of B_k to B_k^θ.

Esscher Transform for Some Specific Models

We consider only one II process ($m = 1$) and consider only constant choices of $\tilde{\theta}$:

1. Time Inhomogeneous Compound Poisson Process.

2. NIG & Hyperbolic Lévy Process->GH.

4. CGMY Lévy process.

Esscher Transform for Some Specific Models: Time Inhomogeneous Compound Poisson Process

Compensator $l(dz, dt) = \lambda(t) F_X(dz) dt$, where F_X is the distribution of the jump size r.v. X and $\lambda(t)$ is the time-dependent jump intensity. The compensator measure onder Q^θ is (Prop. 4.4.):

$$l^\theta(dt, dz) = \lambda(t) e^{\tilde{\theta} z} F_X(dz) dt.$$

A common choice of jump size is the exponential distribution with exapectation μ_J. Then, we find the compensator measure under Q^θ to be

$$l^\theta(dt, dz) = \frac{\lambda(t)}{\mu_J} \exp(-(\frac{1}{\mu_J} - \tilde{\theta}) z) dz dt.$$

Hence, with $\tilde{\theta} < 1/\mu_J, I(t)$ will remain a compound Poisson process under Q^θ, with expectation $1/(1/\mu_J - \tilde{\theta})$ and intensity $\lambda(t)/(1 - \mu_J \tilde{\theta})$.

Esscher Transform for Some Specific Models: NIG and Hyperbolic

NIG and hyperbolic processes are special case of GH Lévy process. From the structure of Lévy measure $\tilde{l}_{GH}()$ of the GH Lévy process, we can see that the structure of the Lévy measure is preserved completely, and the only change is that the skewness parameter β in $\tilde{l}_{GH}(dz)$ is transformed to $\beta + \tilde{\theta}$. Hence, we still have a GH Lévy process with all parameters unchanged except skewness, which is $\beta + \tilde{\theta}$ under Q^{θ}.

We can see that a positive price of jump risk leads to a more right-skewed distribution, and therefore also more emphasis on the bigger jumps after Esscher tranformation.

Esscher Transform for Some Specific Models: CGMY

From the Lévy measure $\tilde{l}_{CGMY}(dz)$ it follows that the G parameter is transformed into $G+\tilde{\theta}$ and the parameter M into $M-\tilde{\theta}$.

Thus, the resulting compensator measure will assign less emphasis on negative jumps, and more on positive ones in the case of $\tilde{\theta} > 0$.

Also, in this case the Esscher tansform preserves the distribution.

Remark. If $\tilde{\theta} \equiv \tilde{\theta}(t)$, then we preserve the distributional properties, and we may say that the resulting II process under Q^{θ} is GH or CGMY distributed for each increment, however, the parameters, of the distribution will now depend on time.

Currency Conversion for Forward and Swap Prices

Sometimes it is convenient to change the denomination of a financial contract from one currency to another. This is a relevant problem for foreign traders in a market.

In the Nord Pool electricity market we have seen a transition from NOK to EUR denominated contracts, and, for instance, Swedish, Danish and Norwegian participants in this market are exposed to currency risk since the contracts are not denominated in their respective kroner.

Currency Conversion for Forward and Swap Prices

There it is most convenient to denominate all contracts in a common currency.

Our currency model is a simple one, where we assume that domestic interest rate and foreign interest rates are both constant. Below we derive the forward exchange rate and the forward commodity price convenience rate.

Note that our focus is on currency conversion so that contracts with prices in different currencies can be consistently converted to a common currency.

Currency Conversion for Forward and Swap Prices

Let Q and Q^* denote the domestic and foreign risk-neutral probability measures, respectively. Domestic and foreign interest rates are assumed to be constants and denoted by r and r^*, respectively. The price at time t of a domestic zero coupon bond with maturity $\tau > t$, denoted $P(t,\tau)$, is defined by

$$P(t,\tau) = E_Q[e^{-\int_t^\tau r ds}|\mathcal{F}_t] = e^{-r(\tau-t)}.$$

Similarly, a foreign zero coupon bond, $P^*(t,\tau)$, is given by

$$P^*(t,\tau) = E_{Q^*}[e^{-\int_t^\tau r^* ds}|\mathcal{F}_t] = e^{-r^*(\tau-t)}.$$

Currency Conversion for Forward and Swap Prices

Let now $X(t)$ be the spot exchange rate prevailing at time t and measured in the ratio

$$\frac{units \quad of \quad domestic \quad currency}{units \quad of \quad foreign \quad currency}.$$

We suppose that $X(t)$ is a positive semimartingale process.

Denote by $f_{FRA}(t, \tau)$ the agreed price at time t for delivery of one unit foreign currency at time τ. This forward contarct is simply called a *forward exchange rate* or *forward exchange rate agreement* (FRA).

Currency Conversion for Forward and Swap Prices

The payoff of a long position at time τ is

$$X(\tau) - f_{FRA}(t, \tau).$$

Under the domestic risk-neutral measure we have (assuming $E_Q[X(\tau)] < +\infty$) that

$$f_{FRA}(t, \tau) = E_Q[X(\tau) | \mathcal{F}_t],$$

since it is costless to enter the FRA.

Currency Conversion for Forward and Swap Prices

Now we use the foreign risk-neutral measure to derive the spot-forward exchange rate realtionship. From a foreign point of view the exchange rate should be replaced by the rate

$$X^*(t) = \frac{1}{X(t)},$$

which is quoted in

$$\frac{units \quad of \quad foreign \quad currency}{units \quad of \quad domestic \quad currency}.$$

Currency Conversion for Forward and Swap Prices

Let $f^*_{FRA}(t, \tau)$ be the forward exchange rate agreement for delivery of $X^*(t)$ at time τ. Using the same line of reasonings as above (and assuming $E_{Q^*}[X^*(\tau)] < +\infty$), we have

$$f^*_{FRA}(t, \tau) = E_{Q^*}[X^*(\tau)|\mathcal{F}_t].$$

The covered interest rate parity gives us the forward exchange rate, $f_{FRA}(t, \tau)$, defined as

$$f_{FRA}(t, \tau) = X(t)e^{(r-r^*)(\tau-t)},$$

and, similarly,

$$f^*_{FRA}(t, \tau) = X^*(t)e^{(r^*-r)(\tau-t)}.$$

Currency Conversion for Forward and Swap Prices: Forwards

Now we consider spot and forward commodity prices in foreign and domestic currency.

Let the spot rate $S(t)$ denote one unit of the commodity in domestic currency.

The price of the same commodity quoted in foreign currency is $S^*(t)$.

No-arbitrage arguments, and assuming no market frictions, give the following spot price relationship

$$S^*(t)X(t) = S(t).$$

Currency Conversion for Forward and Swap Prices: Forwards

Denote by $f(t,\tau)$ the domestic price at time t of a forward contract with delivery at $\tau > t$. We recall that $f(t,\tau) = E_Q[S(\tau)|calF_t]$.

Consider a forward contract for the same commodity, but with the commodity (and the forward contract) denominated in foreign currency.

Let $f^*(t,\tau)$ denote the forward price in foreign currency. Naturally we have the following relationship

$$f^*(t,\tau) = E_{Q^*}[S^*(\tau)|calF_t].$$

Currency Conversion for Forward and Swap Prices: Forwards

The forward price for the commodity denoted in domestic and foreign currency is linked through the exchange rate in the following way

$$f(t,\tau) = f_{FRA}(t,\tau) f^*(t,\tau) = X(t) e^{(r-r^*)(\tau-t)} f^*(t,\tau).$$

The domestic forward price is the forward price converted at the forward exchange rate.

Currency Conversion for Forward and Swap Prices: Swaps

Let us put the same arguments on commodity swap prices in foreign and domestic currency.

Recall that

$$F(t,\tau_1,\tau_2) = E_Q[\int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2)S(u)du|\mathcal{F}_t].$$

Denote by $F^*(t,\tau_1,\tau_2)$ the swap price on the same commodity in foreign currency. Under natural integarbility conditions on $S^*(u)$ we have that

$$F^*(t,\tau_1,\tau_2) = E_{Q^*}[\int_{\tau_1}^{\tau_2} w^*(u,\tau_1,\tau_2)S^*(u)du|\mathcal{F}_t] = \int_{\tau_1}^{\tau_2} w^* f^*(t,u)du,$$

where

$$w^* = \frac{\widehat{w}^*(u)}{\int_{\tau_1}^{\tau_2} \widehat{w}^*(v)dv}.$$

Currency Conversion for Forward and Swap Prices: Swaps

We have

$$w^*(u) = 1$$

if the settlement takes place at the end of the delivery period, while settlement during the delivery period gives

$$w^*(u) = e^{-r^*u}.$$

Currency Conversion for Forward and Swap Prices: Swaps

The proposition below shows how foreign and domestic swap prices arwe related through a change of currency.

Proposition 4.5. Suppose that

$$\begin{aligned} E_Q[\textstyle\int_{\tau_1}^{\tau_2} |w(u,\tau_1,\tau_2)X(u)|du] &< +\infty \quad and \\ E_Q[\textstyle\int_{\tau_1}^{\tau_2} |w(u,\tau_1,\tau_2)S(u)|du] &< +\infty. \end{aligned}$$

Foreign and domestic swap prices are realted in the following way

$$F(t,\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} w(u,\tau_1,\tau_2) f_{FRA}(t,u)du \times F^*(t,\tau_1,\tau_2).$$

Currency Conversion for Forward and Swap Prices: Swaps

Therefore, a swap price can be denoted in foreign currency by converting it with an appropriate weighting of the forward exchange rate over the settlement period of the contract.

Chapter 4: 40 pages; I did 15; What is left? -just only 25 pages

1. Risk-Neutral Forward and Swaps Price Modelling: Risk-Neutral Probabilities and Esscher Transform, Examples-*done*

2. Currency conversion for forward and swap prices-*done*

3. Pricing of Forwards: The Geometric and Arithmetic Cases-*next time*

4. Pricing of Swaps: The Geometric and Arithmetic Cases-*next time*

The End

Thank You for Your Time and Attention!