

# The LIBOR Market Model - The Seminal Papers and Beyond

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# Introduction

# Introduction

## Why interest rate modeling?

- Interest rate derivatives represent **more than 60% of the over-the-counter markets** over the years (in terms of notional amount)
- Biggest difference to other OTC-markets: in interest rate markets there is a **whole family of underlyings** to be modeled, indexed by the time of maturity, rather than a single one
- **Initial approaches:** short rates or instantaneous forward rates
- PROBLEMS:
  - normally distributed dynamics  $\Rightarrow$  negative interest rates with positive probabilities  $\Rightarrow$  market not arbitrage free
  - Log-normally distributed dynamics (or Log-normally distributed bond prices) solve this problem, BUT explode with positive probability
  - don't exist in practice, not observable
  - complicated formulae for caps etc., calibration not satisfying

# Introduction

## Trading volume interest rates

Table 1. Amounts outstanding of over-the-counter derivatives by risk category and instrument (in billions of US dollars)

	December 2006	December 2007	June 2008	December 2008
Foreign exchange	40,271	56,238	62,983	49,753
Interest rate	291,581	393,138	458,304	418,678
Equity-linked	7488	8469	10,177	6494
Commodity	7115	8455	13,229	4427
Credit default swaps	28,650	57,894	57,325	41,868
Unallocated	43,026	71,146	81,708	70,742
<b>Total</b>	<b>418,131</b>	<b>595,341</b>	<b>683,726</b>	<b>591,963</b>

*Source:* BIS Quarterly Review, September 2009.

# Introduction

## New approach

- More recently: effective rates (tradable market rates) such as LIBOR or EURIBOR are modeled directly
- Introduced in 1997 by Brace, Gatarek & Musiela and Miltersen, Sandmann & Sondermann
- IDEA: Change from continuously compounded to simple rates (being Log-normally distributed)
- LIBOR=London InterBank Offered Rate
- **What are the advantages?**
  - Evolution of real market rates, are directly observable in the market → both spot and forward interest rates are quoted as simple rates per annum
  - Advantage: captures the dynamics of the entire curve of interest rates
  - Closed form solutions for interest rate caps and floors as well as puts and calls written on zero-coupon bonds ⇒ Pricing formulas in modification of BS form (esp. easy to calibrate)

# Mathematical Background

# Mathematical Background

## Basic Assumptions

- Filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , where the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is assumed to satisfy the usual conditions
- market arbitrage-free  $\Rightarrow \exists$  (not necessarily unique) risk-neutral measure  $\mathcal{Q}$
- investment horizon  $T^*$  (large, but finite)
- risk-free interest rate given by stochastic process  $(r(t))_{t \geq 0}$
- money market account is  $(b(t))_{t \geq 0}$

$$b(t) = \exp \left( \int_0^t r(s) ds \right)$$

- continuously compounded discount factor  $(\beta(t))_{t \geq 0}$

$$\beta(t) = \exp \left( - \int_0^t r(s) ds \right)$$

# Mathematical Background

## Measures of the Market

### Physical (Subjective) Measure $\mathbb{P}$

- gives actual probabilities of the events
- does not take risk premia into account  $\Rightarrow$  cannot be used for pricing

### Spot Martingale Measure $\mathcal{Q}$

- Measure with respect to which the discounted security price processes become martingales (discounting with respect to  $b$ )
- Risk-neutral pricing: the time- $t$  price of a random payoff  $X$  at time  $T_k$  is given by

$$p(t) = \mathbb{E}_{\mathcal{Q}} \left[ \frac{\beta(T_k)}{\beta(t)} X \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathcal{Q}} \left[ \frac{b(t)}{b(T_k)} X \middle| \mathcal{F}_t \right]$$

- $\Rightarrow \beta(t) p(t) = \mathbb{E}_{\mathcal{Q}} [\beta(T_k) X | \mathcal{F}_t]$  is a  $\mathcal{Q}$ -martingale

# Mathematical Background

## Change of Numéraire Technique

**Theorem 1.** *Let  $X = (X(t))_{t \geq 0}$  be a numéraire.*

- *If the discounted process  $\tilde{X} = (\beta(t) X(t))_{t \geq 0}$  is a  $\mathcal{Q}$ -martingale, then there exists a probability measure  $\mathcal{Q}^X$  on  $(\Omega, \mathcal{F})$ , defined by the Radon-Nikodym derivative*

$$\left. \frac{d\mathcal{Q}^X}{d\mathcal{Q}} \right|_{\mathcal{F}_t} := Z_X(t) := \frac{X(t) \beta(t)}{X(0)}, \quad t \in [0, T]$$

*such that the discounted primary tradet asset prices are  $\mathcal{Q}^X$ -martingales. For every attainable contingent claim  $D(T)$ , its price at time  $t$  is given by*

$$b(t) \cdot \mathbb{E}_{\mathcal{Q}} \left[ \frac{D(T)}{b(T)} \middle| \mathcal{F}_t \right] = X(t) \cdot \mathbb{E}_{\mathcal{Q}^X} \left[ \frac{D(T)}{X(T)} \middle| \mathcal{F}_t \right]$$

- *If  $Y$  is another numéraire with  $\tilde{Y} = (\beta(t) Y(t))_{t \geq 0}$  also being a  $\mathcal{Q}$ -martingale, then for all  $t$*

$$\left. \frac{d\mathcal{Q}^X}{d\mathcal{Q}^Y} \right|_{\mathcal{F}_t} = \frac{X(t) Y(0)}{X(0) Y(t)}$$

# Mathematical Background

## Zero-coupon bond

- **Zero-coupon bond**  $B(t, T)$ : Pays 1 at maturity  $T$

$$B(t, T) = \frac{1}{\beta(t)} \mathbb{E}_{\mathcal{Q}}[\beta(T) | \mathcal{F}_t] \quad \forall 0 \leq t \leq T$$

- In particular  $B(T, T) = 1$  and  $(B(t, T) \beta(t))_{t \geq 0}$  is obviously a  $\mathcal{Q}$ -martingale.
- Since  $B(t, T) > 0$  for all  $t \in [0, T] \Rightarrow$  natural to consider zero-coupon bond price process  $(B(t, T))_{t \geq 0}$  as a numéraire

# Mathematical Background

## $T$ -forward measure $\mathcal{Q}^T$

- **Forward contract:** Delivers one share of an asset  $(S(t))_{t \geq 0}$  at time  $T$  in exchange for a price  $K$ . Value of the contract at time  $0 < t \leq T$

$$\begin{aligned} V(t) &= \frac{1}{\beta(t)} \mathbb{E}_{\mathcal{Q}} [\beta(T) (S(T) - K) | \mathcal{F}_t] \\ &= S(t) - \frac{K}{\beta(t)} \mathbb{E}_{\mathcal{Q}} [\beta(T) | \mathcal{F}_t] \\ &= S(t) - K \cdot B(t, T) \end{aligned}$$

using that  $\beta(t) S(t)$  is a martingale.  $K$  is determined such that the contract is worth 0 at time 0.

$$Forward_S(t, T) = \frac{S(t)}{B(t, T)}$$

- change of numéraire theorem  $\Rightarrow (Forward_S(t, T))_{t \geq 0}$  is a  $\mathcal{Q}^{B(t, T)}$ -martingale.
- $\Rightarrow$  The  $T$ -forward measure  $\mathcal{Q}^{B(t, T)} =: \mathcal{Q}^T$  is used to price payoffs that occur at time  $T$ !

# The Log-normal Model

# The Log-normal Model

## An Overview

- LIBOR market models were introduced in the seminal papers of Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997), and Jamshidian (1997).
- LIBOR rates are modeled as the exponential of a Brownian motion under their corresponding forward measure, hence they are Log-normally distributed.
- $\Rightarrow$  Log-normal LIBOR market model
- Caplets are priced by Black's Formula (accordance with market practice!)

# The Log-normal Model

## Reminder: The HJM model

- Model the instantaneous forward rates  $f$

$$d_t f(t, T) = \mu(t, T) dt + \gamma^*(t, T) dW(t)$$

where  $W$  is a  $\mathcal{Q}$ -Wiener process. Bond prices are given by

$$B(t, T) = \mathbb{E}^{\mathcal{Q}} \left[ \exp \left( - \int_t^T r(u) du \right) \middle| \mathcal{F}_t \right] = \exp \left( - \int_t^T f(t, u) du \right)$$

- Market arbitrage-free if

$$d_t f(t, T) = \gamma^*(t, T) \sigma^*(t, T) dt + \gamma^*(t, T) dW(t)$$

where  $\sigma^*(t, T) = \int_t^T \gamma^*(t, u) du$  is then the volatility of the  $T$ -maturity bond.

- Then

$$dB_t(t, T) = r(t) B(t, T) dt - \sigma^*(t, T) B(t, T) dW(t)$$

- Also

$$\frac{B(t, T)}{b(t)} = B(0, T) \exp \left( - \int_0^t \sigma^*(s, T-s) dW(s) - \frac{1}{2} \int_0^t |\sigma^*(s, T-s)|^2 ds \right)$$

# The Log-normal Model

## Effective Simple Interest Rates

Let  $\delta > 0$  be fixed. Consider investment:

1. At time  $t$  invest \$  $B(t, T + \delta) / B(t, T)$  at time  $T$
2. Payback at  $T + \delta$ : payback of one dollar

Investment Strategy:

1. Go short  $B(t, T + \delta) / B(t, T)$  bonds maturing at time  $T$
2. Go long one bond maturing at time  $T + \delta$

$\Rightarrow$  value of the portfolio at time  $t$

$$-\frac{B(t, T + \delta)}{B(t, T)} B(t, T) + B(t, T + \delta) = 0$$

# The Log-normal Model

## Effective Simple Interest Rates

- Forward LIBOR  $L(t, T, T + \delta)$  = simple forward interest rate of this investment:

$$\frac{B(t, T + \delta)}{B(t, T)} (1 + \delta L(t, T, T + \delta)) = 1$$

$$\Leftrightarrow 1 + \delta L(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)}$$

$$\Rightarrow L(t, T, T + \delta) = \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right) = \frac{\exp \left( \int_T^{T+\delta} f(t, u) du \right) - 1}{\delta}$$

- Connection to forward rates:

$$f(t, t + \tau) = r(t, \tau) = \lim_{\delta \downarrow 0} \frac{\exp \left( \int_T^{T+\delta} f(t, u) du \right) - 1}{\delta}$$

# The Log-normal Model

## Discrete Tenor Market

- Assume a tenor structure  $0 = T_0 < \dots < T_N$ , the constant tenor is  $\delta = T_{k+1} - T_k$
- We assume there  $n$  forward LIBOR rates of the market, where the LIBOR for the time interval  $[T_i, T_{i+1}]$  at time  $t$  is given by  $L_i(t) := L_i(t, T_i, T_{i+1})$  such that

$$1 + \delta L_i(t) = \frac{B(t, T_i)}{B(t, T_{i+1})} = F(t, T_i, T_{i+1}) =: F_i(t)$$

- Observe

$$B(T_i, T_k) = \prod_{j=1}^{k-1} \frac{1}{1 + \delta L_j(T_i)}$$

- Now: Forward prices  $(F_i(t))_t$  are martingales under the  $T_{i+1}$  measure
- $\Rightarrow$   $i$ -th forward LIBOR rate  $L_i$  follows the  $T_{i+1}$ -forward measure  $\mathcal{Q}^{T_{i+1}}$ -dynamics :

$$dL_i(t) = \sigma_i(t) \cdot L_i(t) dW^{T_{i+1}}(t), \quad L_i(0) > 0$$

where  $W^{T_{i+1}}$  is assumed to be a  $\mathcal{Q}^{T_{i+1}}$ -Brownian motion and  $\sigma_i$  is strictly positive, bounded and deterministic.

# The Log-normal Model

## The Dynamics

- The solution of this SDE is given by

$$L_i(t) = L_i(0) \exp \left( \int_0^t \sigma_i(s) dW^{T_{i+1}} - \frac{1}{2} \int_0^t \sigma_i^2(s) ds \right) \quad (0.1)$$

- thus

$$L_i(T_i) = L_i(t) \exp \left( \int_t^{T_i} \sigma_i(s) dW^{T_{i+1}} - \frac{1}{2} \int_t^{T_i} \sigma_i^2(s) ds \right) \quad (0.2)$$

$$=: L_i(t) \exp(X_i(t)) \quad (0.3)$$

where

$$X_i(t) \sim \mathcal{N} \left( -\frac{1}{2} \rho_i^2(t), \rho_i^2(t) \right)$$

and  $\rho_i^2(t) := \int_t^{T_i} \sigma_i^2(s) dW^{T_{i+1}}(s)$ .

# The Log-normal model

Dynamics with respect to the risk-neutral  $\mathcal{Q}$

- Consider the  $i$ -th LIBOR rate under the  $T_{i+1}$ -forward measure  $\mathcal{Q}^{T_{i+1}}$   
 $dL_i(t) / L_i(t) = \sigma_i^L(t) dW^{T_{i+1}}(t)$  with  $W^{T_{i+1}}$  a  $\mathcal{Q}^{T_{i+1}}$  Wiener process
- How do the dynamics look like after the risk-neutral spot martingale measure  $\mathcal{Q}$ ?
- $\Rightarrow$  measure change by the Radon-Nikodym-derivative

$$\left. \frac{d\mathcal{Q}^{T_{i+1}}}{d\mathcal{Q}} \right|_{\mathcal{F}_t} = \frac{B(0, T_{i+1})}{B(t, T_{i+1}) \beta(t)} = \exp \left( - \int_0^t \gamma_{i+1}^*(u) dW^{\mathcal{Q}}(u) - \frac{1}{2} \int_0^t (\gamma_{i+1}^*(u))^2 du \right)$$

- Girsanov's theorem then yields

$$W^{T_{i+1}}(t) = W^{\mathcal{Q}}(t) - \int_0^t \gamma_{i+1}^*(u) du = W^{\mathcal{Q}}(t) - \sigma_{i+1}^*(t)$$

- Thus, the risk-neutral interest dynamics are given by

$$\frac{dL_i(t)}{L_i(t)} = \sigma_i^L(t) \gamma_{i+1}^*(t) dt - \sigma_i^L(t) dW^{\mathcal{Q}}(t)$$

where  $W^{\mathcal{Q}}$  is a  $\mathcal{Q}$ -Brownian motion.

# The Log-normal model

Dynamics with respect to the terminal measure  $\mathcal{Q}^{T_N}$

- Consider the  $i$ -th LIBOR rate under the  $T_{i+1}$ -forward measure  $\mathcal{Q}^{T_{i+1}}$   
 $dL_i(t) / L_i(t) = \sigma_i^L(t) dW^{T_{i+1}}(t)$  with  $W^{T_{i+1}}$  a  $\mathcal{Q}^{T_{i+1}}$  Wiener process
- measure change by the Radon-Nikodym-derivative (change of numéraire theorem)

$$\begin{aligned} \left. \frac{d\mathcal{Q}^{T_i}}{d\mathcal{Q}^{T_{i+1}}} \right|_{\mathcal{F}_t} &:= Z_i(t) = \frac{B(0, T_{i+1})}{B(0, T_i)} \cdot \frac{B(t, T_i)}{B(t, T_{i+1})} = \dots \\ &= \exp \left( \int_0^t \eta_i(s) dW^{T_{i+1}}(s) - \frac{1}{2} \int_0^t \eta_i^2(s) ds \right) \end{aligned}$$

with  $\eta_i(t) = \eta(t, T_i, T_{i+1}) = \frac{\delta L_i(t)}{1 + \delta L_i(t)}$ .

- After again applying Girsanov's theorem, we recursively find

$$\frac{dL_i(t)}{L_i(t)} = - \sum_{k=i+1}^{N-1} \eta_k(t) \sigma_i(t) dt + \sigma_i(t) dW^{T_N}(t)$$

- NOTE: Since  $\eta_k \sigma_i > 0$  for all  $i = 1, \dots, N - 1$ , only the LIBOR  $L_N$  is a martingale under the  $\mathcal{Q}^{T_N}$ -measure

# The Log-normal Model

## Example: Pricing Caplet

- Easy: Determine price of a caplet (for example): A caplet protects its owner by requiring him to pay only the cap  $\delta\kappa$ , if  $\delta L_i(T_i) > \delta\kappa$ . At the end of the period  $[T_i, T_{i+1}]$  the owner of the caplet receives a payment that is equal to the amount in which the interest rate exceeds the prespecified strike price.
- Via risk-neutral evaluation and measure change, we find

$$\begin{aligned}
 \text{Caplet}_i(t) &= \mathbb{E}_{\mathcal{Q}} \left[ \frac{\beta(T_{i+1})}{\beta(t)} \delta \max \{L_i(T_i) - \kappa, 0\} \middle| \mathcal{F}_t \right] \\
 &= \frac{1}{\frac{d\mathcal{Q}}{d\mathcal{Q}^{T_{i+1}}} \bigg|_{\mathcal{F}_t}} \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \frac{d\mathcal{Q}}{d\mathcal{Q}^{T_{i+1}}} \bigg|_{\mathcal{F}_{T_{i+1}}} \frac{\beta(T_{i+1})}{\beta(t)} \delta \max \{L_i(T_i) - \kappa, 0\} \middle| \mathcal{F}_t \right] \\
 &= \delta B(t, T_{i+1}) \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} [(L_i(T_i) - K)^+ | \mathcal{F}_t] \\
 &= \delta B(t, T_{i+1}) \left( L_i(t) \cdot \mathcal{N}(\tilde{d}_1) - \kappa \mathcal{N}(\tilde{d}_2) \right)
 \end{aligned}$$

where  $\mathcal{N}$  denotes the cdf of the standard normal distribution and  $\tilde{d}_1, \tilde{d}_2$  are given a bit differently than the BS model.

# The Log-normal Model

## Example: Pricing Caplet

- An interest rate cap is a derivative in which the buyer receives payments at the end of each period in which the interest rate exceeds the agreed strike price.
- For the cap over the time period  $[0, T_n]$ , with  $T_0 = 0, \dots, T_n = n\delta$ , the cap is a series of payments

$$\delta (L_i (T_i) - \kappa)^+$$

- Thus the value at time  $t$  = value of all remaining caplets

$$Cap(t) = \sum_{i; T_i \geq t} Caplet_i(t)$$

# Extensions of the Log-normal model

# Extensions of the Log-normal model

## Requirements

A model for LIBOR rates should satisfy the following requirements:

1. LIBOR rates should be **non-negative**:  $L_i(t) \geq 0$  for all  $t \in [0, T]$
2. The model should be **arbitrage-free**:  $L_i(t)$  is a  $Q^{T_{i+1}}$ -martingale
3. The model should be **analytically tractable**, easy to implement, and quick to calibrate to market data ( $\rightarrow$  “closed-form” solutions for caps and swaptions)
4. The model should provide a **good calibration to market data** of liquid derivatives (caps and swaptions).

# Extensions of the Log-normal model

## Type of Extensions

- **Lévy processes**, e.g. jump diffusions (see Glasserman and Khou, Belomestny and Schoenmakers)

$$\begin{aligned} \frac{dL_i(t)}{L_i(t)} = & - \sum_{j=i+1}^{n-1} \frac{\delta L_j(t-)}{1 + \delta L_j(t-)} \eta_i^T(t) \eta_j(t) dt + \eta_i^T(t) dW^{T_n} \\ & - \int_E \nu^{T_n}(dt, du) \psi_i(t, u) \left( \prod_{j=i+1}^{n-1} \left( 1 + \frac{\delta L_j(t-)}{1 + \delta L_j(t-)} \psi_j(t, u) \right) - 1 \right) \\ & + \int_E \psi_i(t, u) (\mu - \nu^{T_n})(dt, du), \quad i = 1, \dots, n-1 \end{aligned}$$

with  $\omega \rightarrow \mu(dt, du, \omega)$  random point measures and  $\nu^{T_n}(dt, du, \omega)$  a  $Q^{T_n}$ -compensator.

# Extensions of the Log-normal model

## Type of Extensions

- **Lévy processes**, e.g. jump diffusions (see Glasserman and Khou (2003), Belomestny and Schoenmakers(2010))
- **Stochastic volatility processes**, e.g. Markov switching processes (see Elliott, & Valchev (2004))
- **General semimartingales** (see, e.g. Jamshidian (1999))

# Markov switching model

# Markov switching model

## The Dynamics

- Before, the dynamics of the  $i$ -th LIBOR rate  $L_i(t) := L_i(t, T_i, T_{i+1})$  under the measure  $\mathcal{Q}^{T_{i+1}}$  were given by

$$dL_i(t) = L_i(t) \sigma_i(t) dW^{T_{i+1}}(t)$$

where  $W^{T_{i+1}}$  denotes a standard Brownian motion w.r.t. the forward measure  $\mathcal{Q}^{T_{i+1}}$

- Now, following the proposition of Elliott & Valchev (2004):

$$dL_i(t) = L_i(t) \sigma_i(t, X_t) dW^{T_{i+1}}(t)$$

where  $X = (X_t)_{t \geq 0}$  denotes a continuous time Markov chain taking values in the state space  $E = \{e_1, \dots, e_N\}$ , with  $X$  and  $W$  are independent

# Markov switching model

## Complete Market

- The market model under consideration has two types of random sources:  $W$  and  $X$
- The inclusion of  $X$  makes the underlying market incomplete.
- Nevertheless, the market can be made complete by introducing “switching-cost securities” (Guo (1999))

# Markov switching model

## The Dynamics

- Under the assumptions of independence, it is easy to show then (Itô!) that for all  $0 \leq t \leq T_i$

$$L_i(T_i) = L_i(t) \exp \left[ \int_t^{T_i} \sigma_i(u, X_u) dW^{T_{i+1}}(u) - \frac{1}{2} \int_t^{T_i} \sigma_i^2(u, X_u) du \right]$$

$$=: L_i(t) \exp \left[ Z(T_i) - \frac{1}{2} \rho_i^X(t) \right]$$

where  $Z(T_i) \sim \mathcal{N}(0, \rho_i^X(t))$ , where

$$\rho_i^X(t) := \int_t^{T_i} \sigma_i^2(u, X_u) du$$

- Analogously to before, we show that

$$\frac{dL_i(t)}{L_i(t)} = - \sum_{k=i+1}^{N-1} \gamma_k(t, X_t) \cdot \sigma_i(t, X_t) dt + \sigma_i(t, X_t) dW^{T_N}(t)$$

- As before, this is only a martingale for the  $n$ -th LIBOR rate, since the drift is positive for any other LIBOR rate  $L_i$  for  $1 \leq i \leq n-1$ .

# Markov-switching Model

## Possibilities to model volatility

- Without loss of generality, one may assume that  $e_i$  is the  $i$ -th canonical unit vector

$$\sigma_i^2(u, X_u) = \tilde{\sigma}_i(u)^T X_u = \tilde{\sigma}_i^2(u, e_1)^T X_u^1 + \dots + \tilde{\sigma}_i^2(u, e_N)^T X_u^N(t)$$

where  $\tilde{\sigma}_i^2(u) = (\tilde{\sigma}(u, e_1), \dots, \tilde{\sigma}_i(u, e_N)) \in \mathbb{R}_+^N$  is some deterministic vector depending only on timepoint  $u$  and  $X_t^i$  indicates the vector  $X_t$  taking the value of the vector  $e_i$ .

- Possible Choice: Exponential or Vasicek volatility structure:

$$\tilde{\sigma}_i(t) = \tilde{\sigma}(T_i - t) = \left( \sigma(e_1) e^{-\lambda(e_1)(T_i-t)}, \dots, \sigma(e_N) e^{-\lambda(e_N)(T_i-t)} \right)^T \in \mathbb{R}_+^N$$

# Markov-switching Model

## Pricing of Caplets

- Assume that the filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  is given by  $\mathcal{G}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^W$ .
- The price of a caplet at  $t$  for the LIBOR rate in the interval  $[T_i, T_{i+1}]$ ,  $t < T_i$  is then given by

$$\begin{aligned}
 \text{Caplet}(t, T_i) &= \delta B(t, T_i) \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} [(L_i(T_i) - \kappa)^+ | \mathcal{G}_t] \\
 &= \delta B(t, T_i) \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \left( L_i(t) \exp \left[ Z(T_i) - \frac{1}{2} \rho_i^X(t) \right] - \kappa \right)^+ \middle| \mathcal{G}_t \right] \\
 &=: \delta B(t, T_i) g(t, T_i)
 \end{aligned}$$

- Set

$$\begin{aligned}
 A &:= \left\{ L_i(t) \exp \left[ Z(T_i) - \frac{1}{2} \rho_i^X(t) \right] - \kappa \geq 0 \right\} \\
 &= \left\{ Z(T_i) \geq \frac{1}{2} \rho_i^X(t) + \ln \left( \frac{\kappa}{L_i(t)} \right) \right\} =: \{Z(T_i) \geq a\},
 \end{aligned}$$

# Markov-switching Model

## Pricing of Caplets

- Then

$$\begin{aligned}
 g(t, T_i) &= \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \left( L_i(t) \exp \left[ Z(T_i) - \frac{1}{2} \rho_i^X(t) \right] - \kappa \right)^+ \middle| \mathcal{F}_t^W \vee \mathcal{F}_{T_i}^X \right] \middle| \mathcal{G}_t \right] \\
 &= \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \left( L_i(t) \exp \left[ Z(T_i) - \frac{1}{2} \rho_i^X(t) \right] \right) 1_A \middle| \mathcal{F}_t^W \vee \mathcal{F}_{T_i}^X \right] \middle| \mathcal{G}_t \right] \\
 &\quad + \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \kappa 1_A \middle| \mathcal{F}_t^W \vee \mathcal{F}_{T_i}^X \right] \middle| \mathcal{G}_t \right] \\
 &= \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \left( L_i(t) \exp \left[ Z(T_i) - \frac{1}{2} \rho_i^X(t) \right] \right) 1_A \middle| \mathcal{F}_t^W \vee \mathcal{F}_{T_i}^X \right] \middle| \mathcal{G}_t \right] \\
 &\quad + \kappa \cdot \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \mathbb{Q}^{T_i}(A \middle| \mathcal{F}_t^W \vee \mathcal{F}_{T_i}^X) \middle| \mathcal{G}_t \right] \\
 &= \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ L_i(t) \int_a^\infty \frac{1}{\sqrt{2\pi \rho_i^X(t)}} \exp(y) \exp\left(\frac{1}{2} \rho_i^X(t)\right) \exp\left(-\frac{y^2}{2\rho_i^X(t)}\right) dy \middle| \mathcal{G}_t \right] \\
 &\quad + \kappa \cdot \mathbb{E}_{\mathcal{Q}^{T_{i+1}}} \left[ \int_a^\infty \frac{1}{\sqrt{2\pi \rho_i^X(t)}} \exp\left(-\frac{y^2}{2\rho_i^X(t)}\right) dy \middle| \mathcal{G}_t \right]
 \end{aligned}$$

# Markov-switching Model

## Pricing of Caplets

- This leads us to

$$\begin{aligned} \text{Caplet}(t, T_i) &= \delta B(t, T_i) \cdot \mathbb{E}_{\mathcal{Q}^{T_i}} [L_i(t) \mathcal{N}(d_1) - \kappa \mathcal{N}(d_2) | \mathcal{G}_t] \\ &= \delta B(t, T_i) \int_0^\infty (L_i(t) \mathcal{N}(d_1) - \kappa \mathcal{N}(d_2)) \Psi(s) ds \end{aligned}$$

where  $\Psi$  denotes the conditional density of  $\rho_i^X(t)$  and

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{L_i(t)}{\kappa}\right) + \frac{1}{2}\rho_i^X(t)}{(\rho_i^X(t))} \\ d_2 &= d_1 - \rho_i^X(t) \end{aligned}$$

# Markov-switching Model

## Pricing of Caplets: Determining the conditional Density

- IDEA: determine the conditional density by inverting the Fourier transform of  $\rho_i^X(t)$
- Conditional characteristic function  $\Phi_X(t, T_i, b) : \mathbb{R}^N \rightarrow \mathbb{C}^N$ :

$$\begin{aligned} \Phi_X(t, T_i, b) &= \mathbb{E}_{\mathcal{Q}^{T_i}} \left[ e^{ib\rho_i^X(t)} \middle| \mathcal{F}_t^X \right] \\ &= \mathbb{E}_{\mathcal{Q}^{T_i}} \left[ \exp \left( ib \int_t^{T_i} \sigma_i^2(u, X_u) du \right) \middle| \mathcal{F}_t^X \right] \end{aligned}$$

- Then, if  $|\Phi_X(t, T_i, b)|$  is integrable, the density function  $\Psi(\rho_i^X(t))$  can be determined via inversion of the Fourier transform

$$\Psi(\rho_i^X(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ib\rho_i^X(t)} \Phi_X(t, T_i, b) db$$

# Markov-switching Model

## Pricing of Caplets

- Example: only two states of the world: normal ( $e_1$ ) and excited ( $e_2$ ). Transition matrix:

$$Q = \begin{pmatrix} \lambda_{e_1, e_1} & \lambda_{e_1, e_2} \\ \lambda_{e_2, e_1} & \lambda_{e_2, e_2} \end{pmatrix}$$

- The volatility function is given by
 
$$\sigma_i^2(u, X_u) = \tilde{\sigma}_i^2(u, e_1)^T X_t^1 + \tilde{\sigma}_i^2(u, e_2)^T X_t^2(t)$$
- Set  $Y_s := X_s \exp\left(ib \int_t^s \sigma_i^2(u, X_u) du\right)$ .
- Then, under certain conditions, one can show

$$\mathbb{E}_{\mathcal{Q}^{T_i}} [Y_s | \mathcal{F}_t^X] = \exp\left(-\int_t^s B_u du\right) X_t$$

where  $B_u = Q + ib \cdot \text{diag}(\tilde{\sigma}_i^2(u))$

- Observe that we have  $1_N^T X_s = 1$ . Hence

$$1_N^T \cdot \mathbb{E}_{\mathcal{Q}^{T_i}} [Y_{T_i} | \mathcal{F}_t^X] = 1_N^T \cdot \exp\left(-\int_t^{T_i} B_u du\right) X_t = \Phi_X(t, T_i, b)$$

# Further Extensions of the Model

# Further Extensions

## Defaultable Bonds

### Construction of survival measure

- now: introduce  $T_k$ -survival measure  $\bar{\mathcal{Q}}^{T_k}$  to price **defaultable** payoffs at  $T_k$ , Radon-Nikodym derivative

$$\left. \frac{d\bar{\mathcal{Q}}^k}{d\mathcal{Q}} \right|_{\mathcal{F}_t} := \bar{Z}_k(t) := \frac{\beta(t) 1_{\{\tau > t\}} \bar{B}_k(t)}{\bar{B}_k(0)}$$

which is a non-negative  $\mathcal{Q}$ -martingale with  $\bar{Z}_k(0) = 1$ .

- However: not strictly positive!! jumps to zero at default, i.e.  $I(\tau) = 0 \Rightarrow$  survival measure  $\bar{\mathcal{Q}}^k$  is **not** equivalent to  $\mathcal{Q}$ , but at least absolutely continuous w.r.t.  $\mathcal{Q} \Rightarrow$  Girsanov's theorem still applicable
- Interpretation

$$\begin{aligned} \bar{\mathcal{Q}}^k(\tau \leq T_k) &= \mathbb{E}_{\bar{\mathcal{Q}}^k} [1_{\{\tau \leq T_k\}}] = \mathbb{E}_{\mathcal{Q}} [\bar{Z}_k(T_k) 1_{\{\tau \leq T_k\}}] \\ &= \mathbb{E}_{\mathcal{Q}} \left[ \frac{\beta(t) \bar{B}_k(t)}{\bar{B}_k(0)} 1_{\{\tau > T_k\}} 1_{\{\tau \leq T_k\}} \right] = 0 \end{aligned}$$

## Further Extensions

### Defaultable Bonds

- **default-risk factor** at time  $t$  for maturity  $T_i$ :  $D_i(t) = \frac{\bar{B}_i(t)}{B_i(t)}$ , nice: will be shown that  $D_k(t)$  is the survival probability until  $T_k$  under the  $T_k$ -forward measure  $\mathcal{Q}^{T_k}$
- **forward credit spread**  $S_i(t) = \bar{L}_i(t) - L_i(t)$
- **discrete-tenor forward default intensity**  $H_i(t) = \frac{1}{\delta} \left( \frac{D_i(t)}{D_{k+1}(t)} - 1 \right)$
- $\Rightarrow$  Instead of determining the dynamics for  $\bar{L}_i$  with respect to  $\bar{\mathcal{Q}}^{T_{i+1}}$ , do this for  $H_i$ .

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