Variance Swaps for Mean-reverting Jump-diffusion Models

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April 10th, 2007

Outline

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- Valuation Strategy
- Pricing under Models with Jump-diffusion
- Conclusion & Future Work

Ⅰ Introduction

Variance swaps are forward contracts on future realized stock variance which took off in late 1998 when the LTCM meltdown. It's payoff function at expiration is

$$N(\sigma_R^2 - K_{\text{var}})$$

where

σ_R^2 — the realized stock variance (quoted in annual term) over the life of contract;

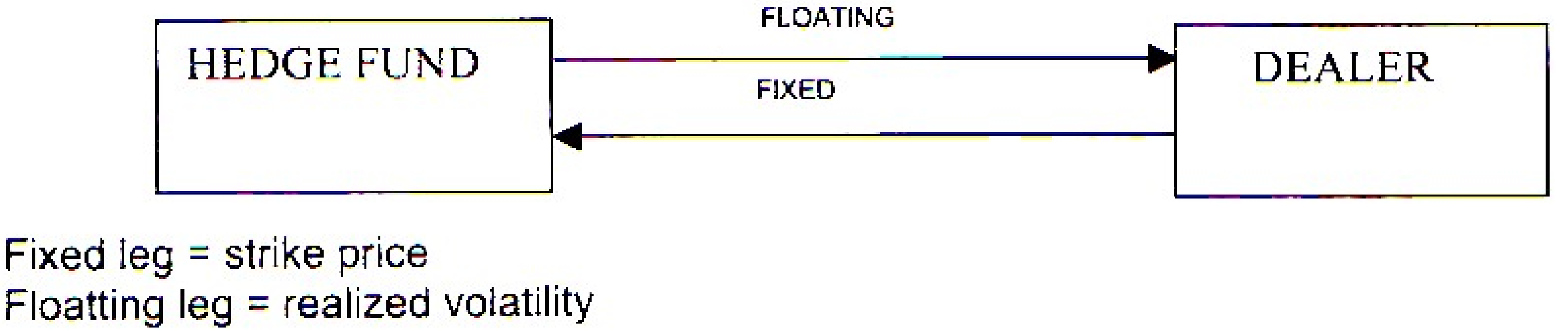
K_{var} — the delivery price for variance;

N — the notional amount of the swap in dollars per annualized volatility point squared.

Volatility swaps are similar contracts on volatility, the square root of variance. It’s payoff function at expiration is

$$N(\sigma_R - K_{vol})$$

How does the Volatility Swap Work?

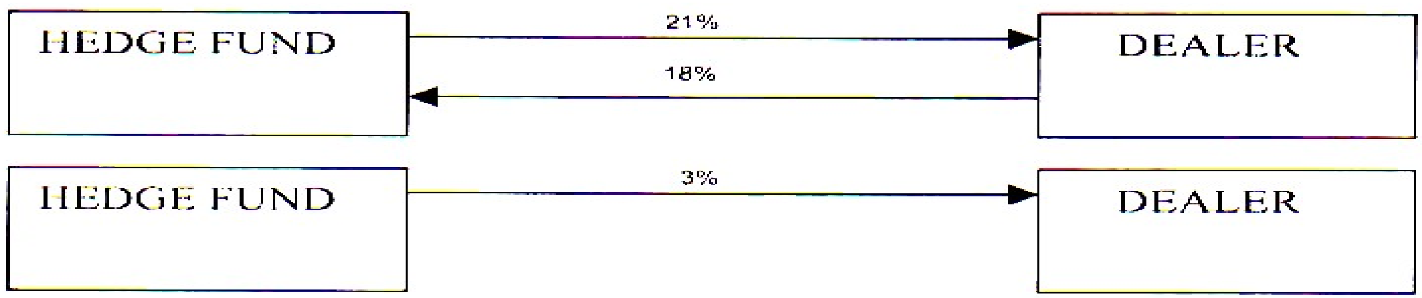


If the realized volatility exceeds the strike price, Hedge Fund pays for the exceeded volatility

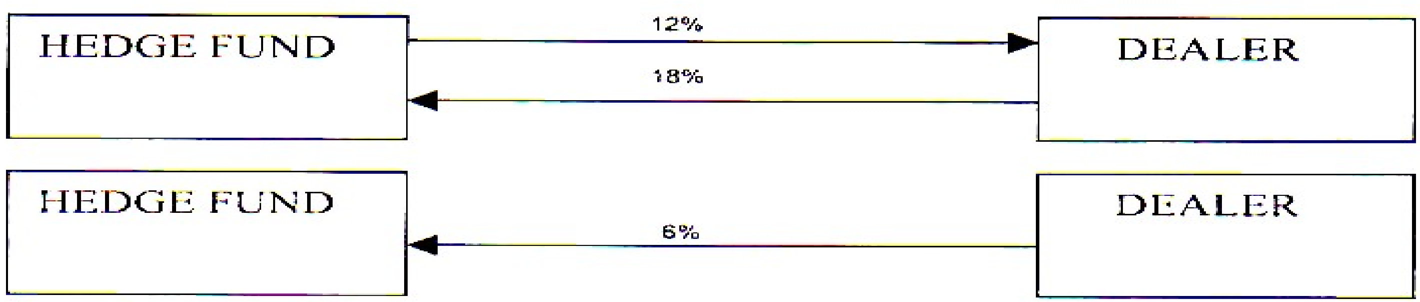
If the strike price exceeds the realized volatility, Dealer pays for the exceeded volatility

SCENARIOS

A – The volatility increases:



B – The volatility decreases:



Ⅱ Valuation Strategy

Valuing variance (volatility) swaps is no different from any other derivative security. For calculating variance swaps, we need to know $E\sigma_R^2$ since the present value of a variance swap is

$$F = E\{e^{-rT} N(\sigma_R^2 - K_{\text{var}})\}$$
$$= e^{-rT} N(E\sigma_R^2 - K_{\text{var}})$$

The present value of a volatility swap can be written as

$$F = E\{e^{-rT} N(\sigma_R - K_{vol})\}$$
$$= e^{-rT} N(E\sigma_R - K_{vol})$$

Therefore, $E\sigma_R$ should be known with

$$E\sigma_R = E(\sqrt{\sigma_R^2}) \approx \sqrt{E(\sigma_R^2)} - \frac{Var(\sigma_R^2)}{8E(\sigma_R^2)^{3/2}}$$

Prove for $E\sigma_R$

By Taylor's Formula,

$$\sqrt{x} = \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0) - \frac{1}{8x_0^{3/2}}(x - x_0)^2 + \cdots$$

Let $x = \sigma_R^2$ and $x_0 = E\sigma_R^2$, we obtain

$$\sqrt{\sigma_R^2} \approx \sqrt{E\sigma_R^2} + \frac{1}{2\sqrt{E\sigma_R^2}}(\sigma_R^2 - E\sigma_R^2) - \frac{1}{8E(\sigma_R^2)^{3/2}}(\sigma_R^2 - E\sigma_R^2)^2$$

$$= \frac{(\sigma_R^2 + E\sigma_R^2)}{2\sqrt{E\sigma_R^2}} - \frac{(\sigma_R^2 - E\sigma_R^2)^2}{8E(\sigma_R^2)^{3/2}}$$

Taking the expectation on both sides, we have

$$E\sigma_R \approx \sqrt{E\sigma_R^2} - \frac{Var(\sigma_R^2)}{8E(\sigma_R^2)^{3/2}}$$

Ⅲ Pricing under Models with Jump-diffusion

General Assumption

(1) Let (Ω, F, F_i, P) be a risk-neutral probability space with filtration $F_t, t \in [0,T]$;

(2) Assume that variance is varying according to stochastic process on the interval $[\tau_i, \tau_{i+1}), i = 0,1,2,\ldots$ while at the random time τ_i , the variance jumps, namely $\sigma_{\tau_i}^2 = J\sigma_{\tau_{i-}}^2$;

(3) N_t , the total number of jumps on the interval $[0, t]$, is a Poisson process with density $\lambda > 0$;

(4) The jump size J satisfy $EJ = 1$ and $\ln J \sim N(-\frac{\sigma_J^{\ 2}}{2}, \sigma_J^{\ 2})$ in model 1 and $EJ = A \neq 1$ in model 2.

Model 1

$$d\sigma_t^2 = k(\alpha - \ln\sigma_t^2)\sigma_t^2 dt + \gamma\sigma_t^2 dW_t + (J-1)\sigma_t^2 dN_t$$

We can get the solution by letting $Y_t = \ln\sigma_t^2$,

$$\sigma_t^2 = \exp\{e^{-kt}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2k})(1-e^{-kt}) + \gamma\int_0^t e^{-k(t-s)} dW_s + \int_0^t e^{-k(t-s)} \ln J dN_s\}$$

Since

$$E(\exp(\gamma\int_0^t e^{-k(t-s)}dW_s)) = \exp[\frac{\gamma^2}{4k}(1-e^{-2kt})]$$

$$E(\exp(\int_o^t e^{-k(t-s)}\ln JdN_s)) = \exp[\lambda\int_0^t (e^{\frac{\sigma_J^2}{2}(e^{-k(t-s)}-\frac{1}{2})^2-\frac{\sigma_J^2}{8}}-1)ds]$$

We have

$$E\sigma_t^2 = \exp[e^{-kt}\ln\sigma_0^2 + (\alpha - \frac{\gamma^2}{2k})(1-e^{-kt}) + \frac{\gamma^2}{4k}(1-e^{-2kt})]\exp[\lambda\int_0^t (e^{\frac{\sigma_J^2}{2}(e^{-k(t-s)}-\frac{1}{2})^2-\frac{\sigma_J^2}{8}}-1)ds]$$

By $E\{\sigma_R^2\} = \frac{1}{T}\int_0^T E\sigma_t^2 dt$, we are done.

Model 2

$$d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dW_t + (J-1)\sigma_t^2 dN_t$$

Since we don't know the solution of this SDE, we can't apply the direct approach in Model 1. Instead, let's consider "jump-to-jump".

By mathematical induction and define $\sum_{i=1}^{0} A^{-i} e^{k\tau_i} = 0$, we reach the following conclusion:

$$E(\sigma_t^2 \mid N_t = m, 0 \le \tau_1 < \tau_2 < \cdots < \tau_m < t)$$

$$= A^m e^{-kt}(\sigma_0^2 - \theta^2) + (A-1)\theta^2 e^{-kt}\left(\sum_{i=1}^{m} A^{m-i} e^{k\tau_i}\right) + \theta^2$$

Therefore,

$$E(\sigma_t^2 \mid N_t = m) = A^m(\sigma_0^2 - \theta^2)e^{-kt} + A^m(A-1)\theta^2 e^{-kt}\left(E\left(\sum_{i=1}^{m} A^{-i} e^{k\tau_i}\right)\right) + \theta^2$$

with

$$E\sum_{i=1}^{m} A^{-i} e^{k\tau_i} = \sum_{i=1}^{m} A^{-i} E e^{k\tau_i}$$

$$= \sum_{i=1}^{m} A^{-i} \frac{m!}{t^m (i-1)!(m-i)!} \int_0^t e^{ks} s^{i-1} (t-s)^{m-i} ds$$

Thus,

$$E\sigma_t^2 = \sum_{m=0}^{\infty} P(N_t = m) E(\sigma_t^2 \mid N_t = m)$$
$$= (\sigma_0^2 - \theta^2) e^{[\lambda(A-1)-k]t} + (A-1)\theta^2 e^{-(\lambda+k)t} G(A,\lambda,t) + \theta^2$$

where

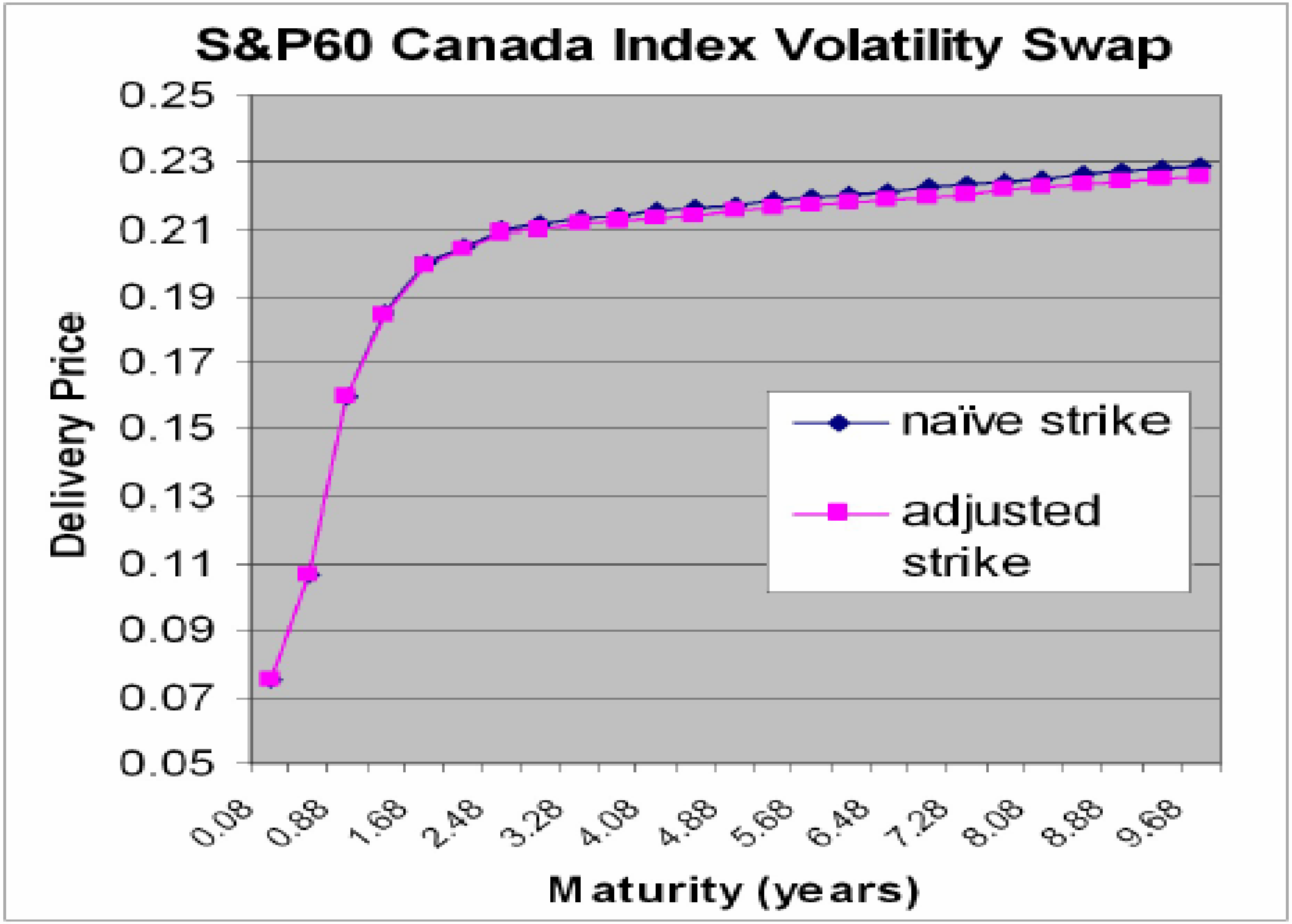
$$G(A,\lambda,t) = \sum_{m=1}^{\infty} (A\lambda)^m [\sum_{i=1}^{m} \frac{1}{A^i (i-1)!(m-i)!} \int_0^t e^{ks} s^{i-1} (t-s)^{m-i} ds]$$

By $E\{\sigma_R^{\ 2}\} = \frac{1}{T}\int_0^T E\sigma_t^2 dt$,we are done.

Ⅳ Conclusion & Future Work

- Formulas derived for variance swaps with different approaches under mean-reverting jump-diffusion models;
- Toy models (data needed)
- Volatility swaps are easier to handle in practice than variance swaps but far more complicated to be valued (the variance of variance required).

Heston Model (No Jumps)



THE END

THANK YOU!