Variance and Volatility Swaps for stochastic volatility models based on Lévy processes

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Levy Processes Definition:

A càdlàg stochastic process $X = \{X_t, t \geq 0\}$ with $X_0 = 0$ a.s. is called a *Lévy process* if the following are satisfied Applebaum [2004]:

1. X has *independent increments*: i.e. for each $n \in \mathbb{N}$ and $0 \geq t_1 \ldots t_{n+1} < \infty$, $\{X_{t_{j+1}} - X_j,\, 1 \leq j \leq n\}$ are independent;
2. *stationary increments*: for any $X_{t_{j+1}} - X_j \stackrel{d}{=} X_{t_{j+1}-t_j} - X_0$;
3. *stochastically continuous*: for all $t > 0$ and $\epsilon > 0$:

$$\lim_{s \longrightarrow t} \mathbf{P}(|X_t - X_s| > \epsilon) = 0.$$

Lévy-Khinchine Formula for Lévy Processes

The characteristic function of X_t is given by $\phi(u) = e^{t\,\eta(u)}$ ($t > 0$ and $u \in \mathbb{R}$), where $\eta(u)$ is known as the characteristic exponent, or Lévy symbol, of the process. The characteristic exponent of L_t can be be expressed as

$$\eta(u) = i\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d - \{0\}} [e^{i\langle u,y\rangle} - 1 - i\langle u, y\rangle \mathbf{1}_{B_1(0)}(y)]\nu(dy),$$

where (b, A, ν) are the characteristics of X_t Applebaum [2004].

Moments and Cumulants of Lévy Processes

Let $\{X_t, t \geq 0\}$ be a Lévy process with Lévy triplet (b, A, ν). The n-th absolute momnt of X_t, $E[|X_t|^n]$ finite for every $t \geq 0$ if and only if $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$. In this case the moments of X_t can be calculated from its characteristic function Cont and Tankov [2004]:

$$E[X_t] = t(b + \int_{|x|\geq 1} x\nu(dx), \tag{1}$$

$$Var[X_t] = t\left(A + \int_{-\infty}^{\infty} x^2\nu(dx)\right), \tag{2}$$

$$c_n(X_t) = t\int_{-\infty}^{\infty} x^n\nu(dx) \quad \text{for } n \geq 3. \tag{3}$$

Examples of Lévy Processes

Familiar examples of Lévy Processes include:

- Standard Brownian motion
- Brownian motion with drift
- The Poisson process
- The compound Poisson Process

Lévy Subordinators

- The *gamma subordinator* is the processes defined as $T = \{T(t), t \geq 0\}$, where each $T(t)$ has a gamma distribution, that is:

$$f_{\Gamma(t)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx},$$

for $x > 0$, is referred to as a *gamma*(a, b) *process* Schoutens [2003].

- The *inverse Gaussian subordinator* is the processes defined as$T = \{T(t), t \geq 0\}$, where each $T(t)$ has an inverse Gaussian distribution, that is:

$$f_{T(t)}(x) = \frac{ct}{x^{3/2}} \exp\left(2ct\sqrt{\pi\lambda} - \lambda x - \frac{\pi c^2 t^2}{x}\right),$$

for $x > 0$ Schoutens [2003].

Subordinated Brownian Motion with Drift

- The variance gamma (VG) process as introduced in Madan and Seneta [1990] results from a stochastic time changed Brownian motion with drift, where a gamma process acts as the time change.
- The The Normal Inverse Gaussian (NIG) Process as introduced in Barndorff-Nielsen [1997] results from a stochastic time changed Brownian motion with drift, where a inverse Gaussian process acts as the time change.

The Lévy-Iô Decomposition

Let X be a Lévy process, then there exists $b \in \mathbb{R}^d$, a Brownian motion W_A with covariance matrix A, and an independent Poisson measure N on $\mathbb{R}_+ \times (\mathbb{R}^d - \{0\})$ such that, for each $t \geq 0$,

$$X_t = bt + W_A(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx),$$

where $\tilde{N}(t, dx) = N(t, dx) - t\nu(dx)$.

Asset price modeling and Derivative Pricing based on Lévy Processes

We closely follow the presentation of the material in Schoutens [2003].

- Let S_t be the price of our stock at time t.
- Assume that $S_t = S_0 e^{mt+L_t}$ where $L = L_t, L \geq 0$ is a Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$.
- Let $V(S_t) = V_t$ be the value of our derivative at time t.
- $V_t = e^{r(T-t)} E_{\mathbf{Q}}[V_T|\mathcal{F}_t]$ where r is the risk free rate of interest and $\mathbf{Q}$ is an equivalent probability measure such that the discounted asset price process, $\{e^{-rt}S_t, t \geq 0\}$, is a martingale.

Equivalent Martingale Measures

- *Esscher transform* see Gerber and Shiu [1994]
- The *mean correcting martingale measure* as presented in Schoutens [2003], is obtained by adding setting the drift term as follows:

$$m = r - \log \phi(-i).$$

where ϕ is the characteristic exponent of the driving Lévy process. This approach is used in the following examples.

Examples of Lévy Process Based Models

- The Black-Scholes Model Black and Scholes [1973]
- The jump diffusion models of Merton [1976] and Kou [2002]
- The variance gamma stock price model of Madan et al. [1998]
- The Normal inverse Gaussian stock price model of Barndorff-Nielsen [1997]
- The Meixner stock price model of Schoutens and Leuven [2001]

Calibration of Lévy Process Option Pricing Based Models

We calculate the implied model parameters in MATLAB for the Black Scholes, VG, NIG and Mexiner models based on the data in Schoutens [2003] (call option prices on the S&P 500 Index at the close of the market on 18 April 2002). For the VG, NIG and Mexiner models we use the FFT pricing method of Carr et al. [1999].

Table: Pricing Errors for Lévy Models

Model	ape	aae	rmse	arpe
NIG	3.9097	2.4129	3.0526	6.2563
Mex	4.1165	2.5405	3.1863	6.78108
VG	4.6964	2.8984	3.7255	6.3573
BS	7.9857	4.9283	3.7255	6.3573

Calibration of Lévy Process Option Pricing Based Models

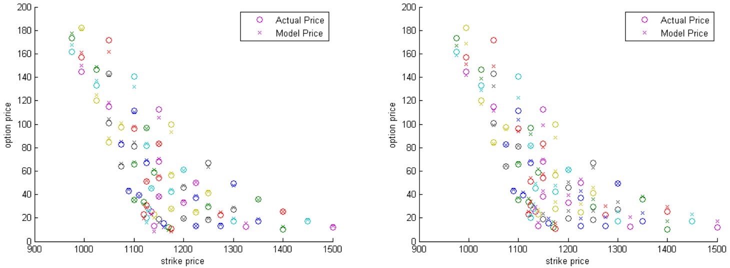

Figure: **NIG and BS Models Calibrated to Market Option Data**

Realized Variance

The following definitions are taken from Demeterfi et al. [1999]. Given a time interval $[0, t]$ and a sample of $N+1$ data points (daily closing prices) $S_0, S_1, \dots, S_N$ with $N = t$ we define the realized variance $v_R^2(t)$ is defined as follows (Cont and Tankov [2004]):

$$v_R^2(t) = \sum_{n=1}^{N} \log^2\left(\frac{S_n}{S_{n-1}}\right). \tag{4}$$

The *annualized* realized variance, $\sigma_R^2(t)$, is defined as:
$\sigma_R^2(t) = \frac{1}{t} v_R^2(t)$

Realized Volatility

The *realized volatility* is defined in a similar way:

$$v_R(t) = \sqrt{\sum_{n=1}^{N} \log^2\left(\frac{S_n}{S_{n-1}}\right)}. \tag{5}$$

The *annualized* realized volatility, $\sigma_R(t)$, is defined as $\sigma_R(t) = \frac{1}{\sqrt{t}} v_R(t)$.

Quadratic Variation and Realized Volatility

We recall that the quadratic variation process of a semimartingale X_t on a time interval $[0, t]$ is the cádlág process defined by Cont and Tankov [2004]

$$[X, X]_t = |X_t|^2 - 2\int_0^t X_{u-}dX_u.$$

Given a mesh $\pi = 0 = t_0, t_1 ... t_{n+1} = t$, we may write the realized variance of S_t over π as

$$v_R^2(\pi) = \sum_{t_i\in\pi} \log(\frac{S_{t_i+1}}{S_{t_i+1}})^2 = \sum_{t_i\in\pi}(X_{t_i+1} - X_{t_i})^2,$$

and thus

$$v_R^2(\pi)\xrightarrow{P}[X, X]_t \quad \text{as} \quad n \to \infty. \tag{6}$$

Quadratic Variation and Realized Volatility

We approximate the expected annualized realized variance using the expected quadratic variation:

$$E\sigma_R^2(T) \approx \frac{1}{T} E[X, X]_T \tag{7}$$

similarly, the expected annualized realized volatility is given by:

$$E\sigma_R(T) \approx \frac{1}{\sqrt{T}} E\sqrt{[X, X]_T} \tag{8}$$

For a detailed investigation of above approximation, see Barndorff-Nielsen and Shephard [2002]. Throughout the rest of the discussion, we will drop the approximation notation and assume that equality holds in (7) and (8).

Implied Volatility

- In the Black Scholes model, if we fix all variables except Black Scholes volatility parameter σ there is a one to one relationship between asset price and σ.
- The *implied volatility* is the value of σ that corresponds the observed option price (for a given strike price and maturity date) Schoutens [2003].
- If we plot the implied volatilities for available option strike prices and maturities we obtain the *implied volatility surface* with its typical skew or smile.

Implied Volatility

We note that in the Black Scholes case the implied volatility and annualized quadratic variation coincide; that is, if X_t is the log returns process of the Black Scholes model then $[X, X]_t = \sigma^2 t$ so the annualized quadratic variation is

$$\frac{1}{T}[X, X]_T = \frac{1}{T}\sigma^2 T = \sigma^2.$$

The concept of Implied Volatility only exists within the context of the Black Scholes model. In section 7.5, Cont and Tankov [2004], and Carr et al. [2005] suggest that the expected quadratic variation should be used as a model independent measure of volatility in non-Gaussian continuous time models.

The CIR Process

The *The-Cox-Ingersoll Ross* (CIR) process is defined by the following stochastic differential equation Schoutens [2003]:

$$dv_t = \kappa(\eta - v_t)dt + \lambda\sqrt{v_t}dW_t,$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion.

Positive OU Processes

The *positive OU process* as presendted in Barndorff-Nielsen and Shephard [2001b] is defined by the following stochastic differential equation:

$$dv_t = -\lambda v_t dt + dZ_{\lambda t}, \tag{9}$$

where $Z = \{Z_t, t \geq 0\}$ is is a Lévy subordinator.
Only positive OU processes will be considered in this presentation.

The gamma-OU Process

If v has a gamma(a,b) marginal law, then the the BDLP Z is a compound Poisson process Schoutens [2003]:

$$z_t = \sum_{n=1}^{N_t} x_n,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter a and $\{x_n, n = 1, \dots\}$ is an iid sequence of of gamma(1,b) random variables. Thus we have

$$E[Z_t] = \frac{at}{b} \tag{10}$$

$$Var[Z_t] = \frac{2at}{b^2} \tag{11}$$

The IG-OU process:

If v has a IG(a,b) marginal law, then the the BDLP Z is the sum of two independent Lévy processes, $Z^{(1)}$, $Z^{(2)}$ that is:

$$Z = \{Z_t^{(1)} + Z_t^{(2)}, t \geq 0\}$$

where $Z^{(1)}$ is an IG(a/2,b) process and $Z^{(2)}$ is a compound Poisson process Schoutens [2003]:

$$z_t = \frac{1}{b^2} \sum_{n=1}^{N_t} x_n^2,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter $\frac{ab}{2}$ and $\{x_n, n = 1, \ldots\}$ is an iid sequence of of N(0,1) random variables.

The IG-OU process:

Thus we have the following:

$$E[Z_t] = \frac{at}{b} \tag{12}$$

$$Var[Z_t] = \frac{2at}{b^3} \tag{13}$$

The Heston Stochastic Volatility Model

The Heston stochastic volatility model Heston [1993] extends the Black Scholes Model by replacing the constant volatility parameter σ with a CIR process.
The risk neutral dynamics of the Heston model are given by:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^{(1)},$$
$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \theta\sigma_t dW_t^{(2)},$$

where where $\{W_t^{(1)}, t \geq 0\}, \{W_t^{(2)}, t \geq 0\}$ are standard Brownian motions with correlation ρ. We note that we may write $W_t^{(1)} = \rho W_t + \sqrt{(}1 - \rho^2) W_t^{(2)}$ where $W_t^{(2)}, W_t$ are independent Brownian motions.

The Bates Stochastic Volatility Model

The risk neutral dynamics of the Bates model Bates [1996] are given by:

$$\frac{dS_t}{S_t} = (r - \lambda\mu_j)dt + \sigma_t dW_t^{(1)} + J_t dN_t,$$
$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \theta\sigma_t dW_t^{(2)},$$

where where $\{W_t^{(1)}, t \geq 0\}, \{W_t^{(2)}, t \geq 0\}$ are standard Brownian motions with correlation ρ, $\{N_t, t \geq 0\}$ is a Poisson process with intensity λ and $\{J_t, t \geq 0\}$ is the jump size distribution, with $EJ_t = \mu_J$ and $\sqrt{\mathrm{var}(1 + \log(J_t))} = \sigma_j$; N_t and J_t are independent, and independent of $W_t^{(1)}$ and $W_t^{(2)}$.

Barndorff-Nielsen and Shepard Model

In Barndorff-Nielsen and Shephard [2001a] it is shown that the risk neutral dynamics of the BNS model may be given by the following:

$$\begin{aligned} S_t &= S_0 e^{X_t} \\ dX_t &= (r - \lambda l(p) - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t + \rho dZ\lambda_t, \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \end{aligned} \tag{14}$$

where $\rho \geq 0, \lambda > 0$, $\{W_t, t \geq 0\}$ is a standard Brownian motion and $\{Z_t, t \geq 0\}$ is a subordinator and $l(p)$ is the Laplace exponent of Z_t.

Barndorff-Nielsen and Shepard Model

Given S_0 and σ_0, the conditional characteristic function of the risk neutral BNS model exists in the following closed form:

$$\begin{aligned}\phi_{X_t}(u) &= \exp(iu(\log s_0 + (r - \lambda l(-\rho))t)) \\ &\times \exp(iu(-\frac{1}{2\lambda}(u^2 + iu)(1 - \exp(-\lambda t))\sigma_0^2)) \\ &\times \exp\left(\lambda \int_0^t l(\rho iu + \frac{1}{2\lambda}(u^2 + iu)(1 - e^{-\lambda(t-s)}))\right).\end{aligned}$$

Barndorff-Nielsen and Shepard Model

We calibrate the IG-BNS model to the data in Schoutens [2003] (call option prices on the S&P 500 Index at the close of the market on 18 April 2002).

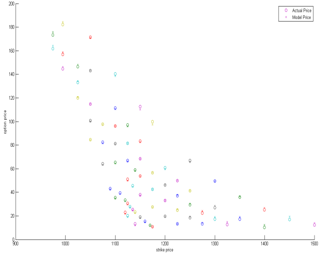

Figure: **Calibration of the IG-BNS Model With Leverage**

Barndorff-Nielsen and Shepard Model

Table: Calibrated Parameters for IG BNS Model

ρ	λ	a	b	σ_0^2
-2.6470	0.8844	0.2125	5.5868	0.0183

Table: Pricing Errors for IG BNS Model

ape	aae	rmse	arpe
1.5531	0.9585	1.1748	3.0642

Time Changed Levy Processes

Let $\{L_t, t \geq 0\}$ be a Lévy process and $V = \{V_t, t \geq 0\}$ be an integrated OU or CIR process. We consider we consider the following process:

$$Y_t = L_{V(t)},$$

We may further extend the above to include the leverage effect, as in Carr et al. [2003], and Kallsen. [2006] as follows:

$$Y_t = L_{V(t)} + \rho Z_t.$$

We adding a drift term m we have the following exponential model:

$$S_t = S_0 \exp(mt + Y_t).$$

The risk neutral dynamics may be found in Schoutens [2003]

CIR Time Change:

CIR Time Change:

$$Y_t = L_{V(t)} + \rho(v_t - v_0), \tag{15}$$
$$dv_t = \kappa(\eta - v_t)dt + \theta\sqrt{v_t}tdZ_t.$$

where L is a Lévy process wither levy triplet (b_L, A_L, ν_L), v is a CIR process driven standard Brownian motion Z with Lévy triplet $(b_Z, A_Z, \nu_Z) = (0, 1, 0)$ and $V_t = \int_0^t v_s ds$; and assumed to be L, independent.

OU Time Change:

OU Time Change:

$$X_t = mt + L_{V(t)} + \rho Z_t, \tag{16}$$

where L is a Lévy process with Lévy triplet (b_L, A_L, ν_L) V is an integrated OU process and with BDLP Z, the Lévy triplet of Z is given by (b_Z, A_Z, ν_Z) ; we assume that L and Z are independent.

Remark

We note that in the models review above, the characteristic function is known in closed form. This allows for option pricing using FFT and hence calibration to option prices, as shown in the case of the BNS model.

Variance Swaps

- A *variance swap* is a forward contract on the annualized realized variance of daily closing prices.
- The payoff at expiry for a variance swap, Demeterfi et al. [1999] to:

$$N(\sigma_R^2(T) - K_{var}).$$

- the value of a variance swap at time t, $V_{\sigma_R^2}(S_t)$, is the present value of expected payoff of the swap in the risk neutral world:

$$V_{\sigma_R^2}(S_t) = Ne^{-r(T-t)}(E[\sigma_R^2(T)|\mathcal{F}_t] - K_{var}). \tag{17}$$

Volatility Swaps

- The payoff at expiry for a volatility swap Demeterfi et al. [1999], is equal to

$$N(\sigma_R(T) - K_{vol}).$$

- The value of a a volatility swap at time t, $V_{\sigma_R}(S_t)$,is the present value of the expected payoff of the swap in the risk-neutral world:

$$V_{\sigma_R}(S_t) = Ne^{-r(T-t)}(E[\sigma_R(T)|\mathcal{F}_t] - K_{vol}). \qquad (18)$$

Volatility Swaps

Since we generally model the variance process σ_t^2, rather than the volatility process, σ_t, the following approximation (Brockhaus and Long [2000])has often been used when pricing volatility swaps:

$$E\sqrt{\sigma_R^2} \approx \sqrt{E\sigma_R^2} - \frac{Var[\sigma_R^2]}{8(E\sigma_R^2)^{3/2}}. \tag{19}$$

A derivation of the above approximation may be found in Javaheri [2004]. We note that the approximation is based on a Taylor expansion, and does not depend on the choice of asset price model.

Remark

Since no money is usually exchanged when the contract is written, to be arbitrage free the present value of the expected payoff should be equal to zero, in both variance and volatility swap contracts, i.e.

$$e^{-r(T)}E[N(\sigma_R^2(T)-K_{var})|\mathcal{F}_0]=0 \Rightarrow E[\sigma_R^2(T)|\mathcal{F}_0]-K_{var}=0,$$

similarly,

$$e^{-r(T)}E[N(\sigma_R(T)-K_{vol})|\mathcal{F}_0]=0 \Rightarrow E[\sigma_R(T)|\mathcal{F}_0]-K_{vol}=0.$$

Thus the fair strike price of the swap should be equal to the expected value of the the realized variance, $K_{var}=E[\sigma_R^2(T)|\mathcal{F}_0]$ and $K_{vol}=E[\sigma_R(T)|\mathcal{F}_0]$ Demeterfi et al. [1999].

Variance and Volatility Swaps in The BNS Model Without Leverage

If we set the leverage parameter $\rho = 0$ in the BNS model (14) we have the following:

$$S_t = S_0 e^{X_t} \tag{20}$$

$$dX_t = (r - \lambda l(p) - \frac{1}{2}\sigma_t^2)dt + \sigma_t S_t dW_t^{(1)}, \tag{21}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0,$$

where $\lambda > 0$, $\{W_t, t \geq 0\}$ is a standard Brownian motion and $\{Z_t, t \geq 0\}$ is a subordinator.

Variance Swaps in The BNS Model Without Leverage

Variance Swaps

We may determine $E[\int_0^T \sigma_s^2 \,|\, \mathcal{F}_0]$ using the characteristic function of the integrated OU process as in Barndorff-Nielsen and Shephard [2003] and Schoutens [2003], thus we have the following:

$$K_{var} = E[\sigma_R^2(s) \,|\, \mathcal{F}_0] = \frac{1 - exp(-\lambda)}{\lambda T}\sigma_0^2 + \frac{\lambda t - 1 - \exp(-\lambda)}{\lambda T}E[z_1]. \tag{22}$$

Volatility Swaps in The BNS Model Without Leverage

Volatility Swaps

In order to apply the Brockhaus-Long approximation (19), we need an expression for the variance of the realized variance, that is $Var[\sigma_R^2 \,|\, \mathcal{F}_0]$. We note that

$$Var[\sigma_R^2 \,|\, \mathcal{F}_0] = \frac{1}{T^2} Var\left[\int_0^T \sigma_s^2 \,|\, \mathcal{F}_0\right],$$

which may again be determined using the characteristic function of the integrated OU process $\int_0^T \sigma_s^2$ Barndorff-Nielsen and Shephard [2003].

Volatility Swaps in The BNS Model Without Leverage

Thus we have:

$$Var[\sigma_R^2 \,|\, \mathcal{F}_0] = \frac{Var[Z_1]}{(T\lambda)^2}\left(\lambda T - 2(1 - e^{-\lambda T}) + \frac{1}{2}(1 - e^{-2\lambda T})\right).$$

We may substitute the above expression into the the Brockhaus-Long approximation (19) in order to determine the fair volatility swap strike price.

Volatility Swaps in The BNS Model Without Leverage

We plot variance and volatility strike prices for an IG BNS model with the following parameters: based on the parameters in 6 (but setting $\rho = 0$):

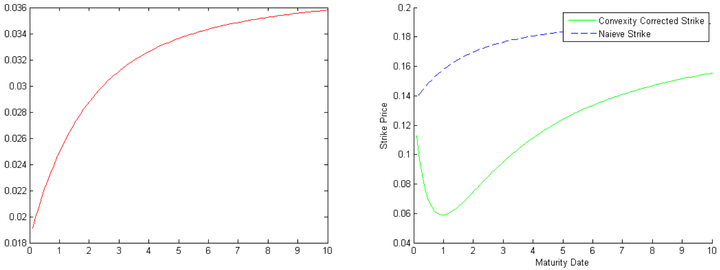

Figure: **Variance and Volatility Swap Strike Prices for IG-BNS Without Leverage**

Realized Variance for Models With Compound Poisson Asset Price Jumps

Consider a log returns process of the following form:

$$dX_t = mdt + \sigma_t W_t + J_t dN_t, \tag{23}$$

Where $\{W_t, t \geq 0\}$, is a standard Brownian motion, $\{\sigma_t^2, t \geq 0\}$ is a CIR or OU process, $\{N_t, t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$ and $\{J_t, t \geq 0\}$ is the jump size distribution, with $EJ_t = \mu_J$ and $\sqrt{\mathrm{var}(1 + \log(J_t))} = \sigma_J$; N_t, J_t, W_t and σ_t^2 are all independent. In this case, due to the presence of the jump process $\{J_t, t \geq 0\}$ in the log returns, we may no longer take σ_t to be the volatility in the Black Scholes sense.

Realized Variance for Models With Compound Poisson Asset Price Jumps

If we assume the log returns process has jumps the form (23) then the realized volatility may be calculated as thus as in Broadie and Jain [2008]

$$[X,X]_t = \int_0^T \sigma_t^2 dt + \sum_{j=1}^{N_t} {\gamma_j}^2,$$

and hence

$$E\left[\sigma_R^2(T)\,|\,\mathcal{F}_0\right] = \frac{1}{T}E\left[\int_0^T \sigma_t^2 dt\,|\,\mathcal{F}_0\right] + \frac{1}{T}\lambda E\gamma^2. \tag{24}$$

This approach has been applied to the Bates model in Broadie and Jain [2008].

Expected QV for Time Changed Models:

In order to calculate the expected realized variance for asset prices models with jumps, other than the compound Poisson case, and to incorporate the leverage effect, we consider the results of Kallsen et al. [2009]:
If X is the log returns process of a time changed SV model of the form (16),(15) and if $\int_0^t E(v_s)ds < \infty$, $\int_{-\infty}^{-\infty} x^2 \nu_L(dx) < \infty$ and $\int_{-\infty}^{-\infty} x^2 \nu_Z(dx) < \infty$, then:

$$E([X,X]_T \mid \mathcal{F}_t) = \Phi_0 + \Phi_1 v_t + [X,X]_t. \tag{25}$$

where Φ_1, Φ_2 are as defined as follows.

CIR Time Change:

$$\Phi_1(t) = \frac{1 - e^{-\kappa(T-t)}}{\kappa} \left(\sigma^2\rho^2 + Var[L_1]\right),$$

$$\Phi_0(t) = \frac{\eta(e^{-\kappa(T-t)} - 1 + \kappa(T - t))}{\kappa} \left(\sigma^2\rho^2 + Var[L_1]\right).$$

OU Time Change:

$$\Phi_1(t) = \frac{1 - e^{-\lambda(T-t)}}{\lambda} Var[L_1],$$

$$\Phi_0(t) = \frac{e^{-\lambda(T-t)} - 1 + \lambda(T-t)}{\lambda^2} (Var[L_1]) (E[Z_1]) + (T-t)\rho^2 Var[Z_1].$$

Variance Swaps in Heston's Model with Leverage

We may now generalize the result in Swishchuk [2004] to the case where for the case where the volatility processes is correlated with asset prices.

In this case L is a standard Brownian motion, thus $(b_L, A_L, \nu_L) = (0, 1, 0)$ The realized variance at time t becomes

$$\begin{aligned} E([X,X]_T \mid \mathcal{F}_t) =& \Phi_0 + \Phi_1 v_t + [X,X]_t, \\ =& (\sigma^2\rho^2 + 1)\left(\frac{1 - e^{-\kappa(T-t)}}{\kappa}(v_t - \eta) + \eta(T-t)\right) \\ &+ [X,X]_t. \end{aligned}$$

Variance Swaps in Heston's Model with Leverage

Thus the fair strike price for a variance swap is given by:

$$K_{var} = (\sigma^2\rho^2 + 1)\left(\frac{1 - e^{-\kappa(T)}}{T\kappa}(v_0 - \eta) + \eta\right). \qquad (26)$$

If we set $\rho = 0$ this reduces to the usual result for the Heston model with independent volatility.

Variance Swaps in the Bates Model with Leverage

We consider the following version of the Bates model:

$$dX_t = \mu dt + \sqrt{v_t} W_t^{(1)} + \rho dv_t + J_t dN_t,$$
$$dv_t = \kappa(\eta - v_t)dt + \theta\sqrt{v_t} t dW_t,$$

where L is a Lévy process, v is a CIR process and $V_t = \int_0^t v_s ds$, $\{N_t, t \geq 0\}$ is a Poisson process with intensity λ and $\{J_t, t \geq 0\}$ is the jump size distribution, with $EJ_t = \mu_J$; N_t, J_t, $W_t^{(1)}$ and W_t are independent.

In this case the leverage effect is included through the term ρdv_t.

Variance Swaps in the Bates Model with Leverage

The QV of the log returns process is given by:

$$[X, X]_t = [X_c, X_c]_t + \sum_{j=1}^{N_t} {\gamma_j}^2,$$

taking expectations and applying (26) as in Broadie and Jain [2008], results in the following expression for the variance swap strike price:

$$K_{var} = (\sigma^2 \rho^2 + 1) \left(\frac{1 - e^{-\kappa(T)}}{T\kappa} (v_0 - \eta) + \eta \right) + \frac{1}{T} \lambda E \gamma^2.$$

Variance Swaps in the BNS Model with Leverage

In the OU time change model with leverage, if we choose L to be a standard Brownian motion, we have the BNS model:

$$
\begin{aligned}
E([X,X]_T \mid \mathcal{F}_t) =& \Phi_0 + \Phi_1 v_t + [X,X]_t, \\
=& \frac{e^{-\lambda(T-t)} - 1 + \lambda(T-t)}{\lambda^2} (E[Z_1]) \\
& + (T-t)\rho^2 Var[Z_1] + \frac{1 - e^{-\lambda(T-t)}}{\lambda} v_t \\
& + [X,X]_t.
\end{aligned}
$$

Variance Swaps in the BNS Model with Leverage

If v has a IG(a,b) marginal law, then we have the following:

$$K_{var} = \frac{e^{-\lambda(T)} - 1 + \lambda(T)}{T\lambda^2}\left(\frac{a}{b}\right) + \rho^2\frac{2a}{b^3} + \frac{1 - e^{-\lambda(T)}}{T\lambda}v_0.$$

For example with a maturity date of 6 months we obtain $K_{var} = 0.0401$, based on the parameters in 6

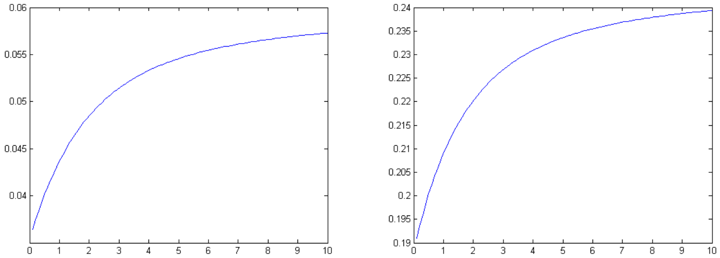

Figure: **Variance Swap and "'Naive"' Volatility Swap Strike Prices for IG-BNS With Leverage**

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