

American options with time-varying parameters via Mellin transforms

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Outline

- ▶ Introduction
- ▶ Review: Mellin transforms and European options
- ▶ American options via Mellin transforms
- ▶ Conclusion

Options

An option is a contract giving the holder the right, but not the obligation, to buy or sell an asset for a prescribed amount (strike price) at, or before, a prescribed future time (expiry).

Common types: call and put

Classification: European and American

How much should the holder pay for this right, i.e., what is the fair value of an option?

Black-Scholes formulas for European options

$$p_e = Ke^{-r(T-t)}N(-d_2) - xe^{-q(T-t)}N(-d_1)$$

$$c_e = xe^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log(x/K) + (r - q + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{\log(x/K) + (r - q - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

p_e	put option price	c_e	call option price
x	underlying asset price	t	time
r	risk-free rate	q	dividend yield
σ	volatility	K	strike price
T	expiry	N	normal distribution function

Black-Scholes PDE and the put-call parity

Black-Scholes PDE for a vanilla option $v(x, t)$

$$(\mathcal{L}v)(x, t) = \frac{\partial v}{\partial t} + \frac{\sigma(t)^2}{2} x^2 \frac{\partial^2 v}{\partial x^2} + [r(t) - q(t)] x \frac{\partial v}{\partial x} - r(t)v = 0$$

Final-value problems for the put and call

$$(\mathcal{L}p_e)(x, t) = 0, \quad p_e(x, T) = \max(K - x, 0)$$

$$(\mathcal{L}c_e)(x, t) = 0, \quad c_e(x, T) = \max(x - K, 0)$$

Put-call parity

$$c_e(x, t) - p_e(x, t) = x e^{-\int_t^T q(\tau) d\tau} - K e^{-\int_t^T r(\tau) d\tau}$$

Plots of European options

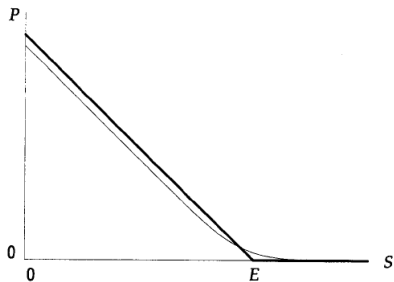


Figure: Put

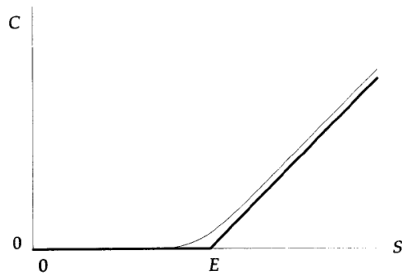


Figure: Call

American options

Formulations

- ▶ Free boundary problem: McKean (1965), van Moerbeke (1976)
- ▶ Linear complementarity problem: Dewynne, et al. (1993)
- ▶ Optimal stopping problem: Jacka (1991), Myneni (1992), Karatzas (1998)

Optimal exercise price = critical asset value marking the boundary between exercise and continuation regions (not known in advance)

Early exercise premium = difference between American and European option values

American options (cont'd)

Analytical approximation methods for the early exercise premium: Geske-Johnson (1984), Barone-Adesi-Whaley (1987), Allegretto, et al. (1995)

Numerical methods

- ▶ Dynamic programming and binomial trees: Cox, et al. (1979), Hull-White (1988), Boyle (1988)
- ▶ Finite difference methods: Brennan-Schwartz (1978), Courtadon (1992), Zhao, et al. (2007)
- ▶ Monte Carlo technique: Longstaff-Schwartz (2001)
- ▶ Method of lines: Meyer-van der Hoek (1997)
- ▶ Randomization method: Carr (1998)

Mellin transforms in option valuation

Panini-Srivastav (2004)

- ▶ American put with constant parameters
- ▶ Inverse Mellin transform leading to complex integrals

Jódar, et al. (2005)

- ▶ European options with constant parameters
- ▶ Inverse Mellin transform leading to complex integrals

Rodrigo-Mamon (2007)

- ▶ European options with time-varying parameters
- ▶ Convolution property leading to real integrals
- ▶ Existence and uniqueness results for general contingent claims

Mellin transform and its properties

Mellin transform of $f : (0, \infty) \rightarrow \mathbb{R}$

$$\hat{f}(\xi) = \int_0^{\infty} x^{\xi-1} f(x) dx$$

Derivative properties

$$\widehat{xf'}(\xi) = -\xi\hat{f}(\xi), \quad \widehat{x^2f''}(\xi) = (\xi + \xi^2)\hat{f}(\xi)$$

Convolution of two functions

$$(f * g)(x) = \int_0^{\infty} \frac{1}{y} f\left(\frac{x}{y}\right) g(y) dy$$

Convolution property

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

Black-Scholes-Mellin equation

Define

$$\hat{v}(\xi, t) = \int_0^\infty x^{\xi-1} v(x, t) dx$$

Apply the Mellin transform to $(\mathcal{L}v)(x, t) = 0$

$$\frac{\partial \hat{v}}{\partial t} + \frac{\sigma(t)^2}{2}(\xi + \xi^2)\hat{v} - [r(t) - q(t)]\xi\hat{v} - r(t)\hat{v} = 0$$

Black-Scholes-Mellin equation

$$\frac{\partial \hat{v}}{\partial t} - \phi(\xi, t)\hat{v} = 0, \quad \phi(\xi, t) = r(t) + \left[r(t) - q(t) - \frac{\sigma(t)^2}{2} \right] \xi - \frac{\sigma(t)^2}{2} \xi^2$$

Black-Scholes kernel and its properties

Assume $x, y > 0$ and $0 \leq t < u \leq T$

Auxiliary functions

$$z_1(x, t, u) = \frac{1}{\left[\int_t^u \sigma(\tau)^2 d\tau\right]^{1/2}} \log x + \frac{\int_t^u [r(\tau) - q(\tau) + \sigma(\tau)^2/2] d\tau}{\left[\int_t^u \sigma(\tau)^2 d\tau\right]^{1/2}}$$

$$z_2(x, t, u) = \frac{1}{\left[\int_t^u \sigma(\tau)^2 d\tau\right]^{1/2}} \log x + \frac{\int_t^u [r(\tau) - q(\tau) - \sigma(\tau)^2/2] d\tau}{\left[\int_t^u \sigma(\tau)^2 d\tau\right]^{1/2}}$$

Black-Scholes kernel

$$\mathcal{K}(x, t, u) = \frac{e^{-\int_t^u r(\tau) d\tau}}{\left[\int_t^u \sigma(\tau)^2 d\tau\right]^{1/2}} N'(z_2(x, t, u)) = \frac{x e^{-\int_t^u q(\tau) d\tau}}{\left[\int_t^u \sigma(\tau)^2 d\tau\right]^{1/2}} N'(z_1(x, t, u))$$

Black-Scholes kernel and its properties (cont'd)

Mellin transform of the Black-Scholes kernel

$$\hat{\mathcal{K}}(\xi, t, u) = e^{-\int_t^u \phi(\xi, \tau) d\tau}, \quad \phi(\xi, \tau) = r(\tau) + \left[r(\tau) - q(\tau) - \frac{\sigma(\tau)^2}{2} \right] \xi - \frac{\sigma(\tau)^2}{2} \xi^2$$

Useful integrals

$$\int \mathcal{K} \left(\frac{x}{y}, t, u \right) dy = -x e^{-\int_t^u q(\tau) d\tau} \left[N \left(z_1 \left(\frac{x}{y}, t, u \right) \right) - \frac{1}{2} \right]$$
$$\int \frac{1}{y} \mathcal{K} \left(\frac{x}{y}, t, u \right) dy = -e^{-\int_t^u r(\tau) d\tau} N \left(z_2 \left(\frac{x}{y}, t, u \right) \right)$$

Valuating European options using Mellin transforms

(i) Consider

$$(\mathcal{L}p_e)(x, t) = 0, \quad p_e(x, T) = \max(K - x, 0)$$

(ii) Obtain the Black-Scholes-Mellin equation

$$\frac{\partial \hat{p}_e}{\partial t} - \phi(\xi, t) \hat{p}_e = 0, \quad \hat{p}_e(\xi, T) = \frac{K^{\xi+1}}{\xi(\xi+1)}$$

(iii) Solution of the linear first-order ODE

$$\hat{p}_e(\xi, t) = e^{-\int_t^T \phi(\xi, \tau) d\tau} \frac{K^{\xi+1}}{\xi(\xi+1)} = \hat{\mathcal{K}}(\xi, t, T) \frac{K^{\xi+1}}{\xi(\xi+1)}$$

(iv) Use the convolution property

$$p_e(x, t) = \int_0^\infty \frac{1}{y} \hat{\mathcal{K}}\left(\frac{x}{y}, t, T\right) \max(K - y, 0) dy$$

(v) Recall the useful integrals to simplify and recover the Black-Scholes formula

(vi) Use the put-call parity for pricing the European call

Valuating European options using Mellin transforms (cont'd)

Rodrigo-Mamon (2007)

- ▶ Used the Mellin transform to obtain the formal European option price for a general payoff function
- ▶ Proved existence for bounded continuous payoffs (e.g., put)
- ▶ Derived a generalized put-call parity
- ▶ Proved existence for payoffs “close” to a bounded continuous function (e.g., call)
- ▶ Derived a maximum principle to show uniqueness

Free boundary problem for American put

Let $x^*(t)$ denote the put optimal exercise price

For $0 < x < x^*(t)$ and $0 \leq t < T$: early exercise is optimal so
 $p_a(x, t) = K - x$

For $x > x^*(t)$ and $0 \leq t < T$: early exercise is not optimal so
 $(\mathcal{L}p_a)(x, t) = 0$

Combine into a single PDE

$$(\mathcal{L}p_a)(x, t) = [q(t)x - Kr(t)]H(x^*(t) - x) \quad (x \neq x^*(t), 0 \leq t < T)$$

Final condition

$$p_a(x, T) = \max(K - x, 0) \quad (x > 0)$$

Boundary (or smooth pasting) conditions

$$p_a(x^*(t), t) = K - x^*(t), \quad \frac{\partial p_a}{\partial x}(x^*(t), t) = -1 \quad (0 \leq t < T)$$

Free boundary problem for American call

Let $y^*(t)$ denote the call optimal exercise price

For $x > y^*(t)$ and $0 \leq t < T$: early exercise is optimal so $c_a(x, t) = x - K$

For $0 < x < y^*(t)$ and $0 \leq t < T$: early exercise is not optimal so $(\mathcal{L}c_a)(x, t) = 0$

Combine into a single PDE

$$(\mathcal{L}c_a)(x, t) = [Kr(t) - q(t)x]H(x - y^*(t)) \quad (x \neq y^*(t), 0 \leq t < T)$$

Final condition

$$c_a(x, T) = \max(x - K, 0) \quad (x > 0)$$

Boundary (or smooth pasting) conditions

$$c_a(y^*(t), t) = y^*(t) - K, \quad \frac{\partial c_a}{\partial x}(y^*(t), t) = 1 \quad (0 \leq t < T)$$

Plots of American options

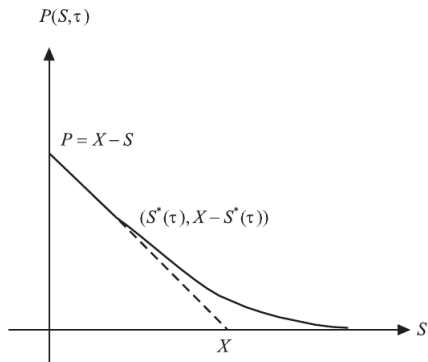


Figure: Put

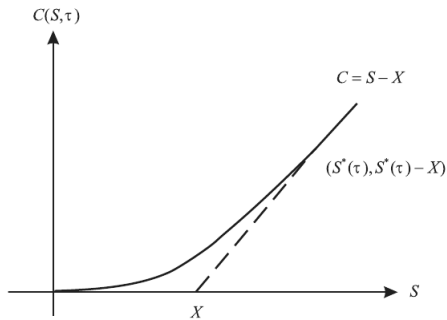


Figure: Call

Valuation of American put

Explicit put formula for arbitrary $x^*(t)$

$$p_a(x, t) = p_e(x, t) + K \int_t^T r(u) e^{-\int_t^u r(\tau) d\tau} N\left(-z_2\left(\frac{x}{x^*(u)}, t, u\right)\right) du \\ - x \int_t^T q(u) e^{-\int_t^u q(\tau) d\tau} N\left(-z_1\left(\frac{x}{x^*(u)}, t, u\right)\right) du$$

Solves PDE and final condition only

If $q \equiv 0$, then

$$p_a(x, t) > p_e(x, t) \quad (0 \leq t < T),$$

i.e., early exercise always optimal for American put for non-dividend-paying assets

Valuation of American put (cont'd)

Alternative expression (after lots of algebra!)

$$\begin{aligned} p_a(x, t) = & (K - x) + x \left[\frac{\partial p_a}{\partial x}(x, t) + 1 \right] \\ & + \int_t^T [Kr(u) - x^*(u)q(u)] \mathcal{H} \left(\frac{x}{x^*(u)}, t, u \right) du \\ & - K \int_t^T r(u) e^{-\int_t^u r(\tau) d\tau} N \left(z_2 \left(\frac{x}{x^*(u)}, t, u \right) \right) du \\ & - K e^{-\int_t^T r(\tau) d\tau} N \left(z_2 \left(\frac{x}{K}, t, T \right) \right) \end{aligned}$$

Evaluate at $x = x^*(t)$ and use both BCs

Valuation of American put (cont'd)

Integral equation for the put optimal exercise price utilizing both BCs

$$\begin{aligned} 0 = & \int_t^T \left[\frac{x^*(u)q(u)}{K} - r(u) \right] \mathcal{K} \left(\frac{x^*(t)}{x^*(u)}, t, u \right) du \\ & + \int_t^T r(u) e^{-\int_t^u r(\tau) d\tau} N \left(z_2 \left(\frac{x^*(t)}{x^*(u)}, t, u \right) \right) du \\ & + e^{-\int_t^T r(\tau) d\tau} N \left(z_2 \left(\frac{x^*(t)}{K}, t, T \right) \right) \end{aligned}$$

Too hard to solve!

Forget about the integral equation and derive an ODE instead

Valuation of American put (cont'd)

Assume r , q , σ are constants with $r > q$: $x^*(T) = K \min(r/q, 1) = K$

Formal ODE for the optimal exercise price

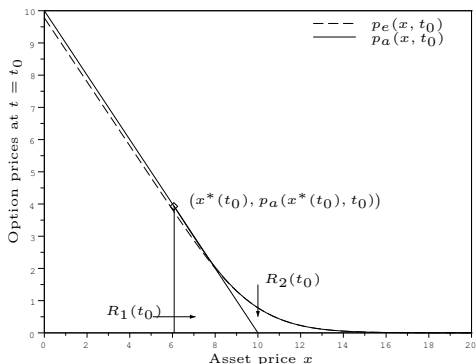
$$0 = \frac{d}{dt} \int_{x^*(t)}^{\infty} p_a(x, t) dx + (\sigma^2 - 2r + q) \int_{x^*(t)}^{\infty} p_a(x, t) dx \\ + [K - x^*(t)] \dot{x}^*(t) - \left(\frac{\sigma^2}{2} - r + q \right) x^*(t)^2 + K(\sigma^2 - r + q)x^*(t)$$

Find an expression for

$$\int_{x^*(t)}^{\infty} p_a(x, t) dx$$

Valuation of American put (cont'd)

Let $R(t)$ be the region below $p_a(x, t)$ from $x = x^*(t)$ to $x = \infty$, $R_1(t)$ the isosceles right triangle inside $R(t)$ of length $K - x^*(t)$, and $R_2(t) = R(t) \setminus R_1(t)$



Valuation of American put (cont'd)

$$\int_{x^*(t)}^{\infty} p_a(x, t) dx = \text{Area of } R(t) = \text{Area of } R_1(t) + \text{Area of } R_2(t)$$

Define

$$\rho(t) = \frac{\text{Area of } R_2(t)}{\text{Area of } R_1(t)}$$

$$\int_{x^*(t)}^{\infty} p_a(x, t) dx = \frac{1}{2} [\rho(t) + 1] [K - x^*(t)]^2$$

First approximation: $\rho(t) \simeq \rho$, $0.2 < \rho < 0.3$ (heuristic)

Approximating ODE for the optimal exercise price

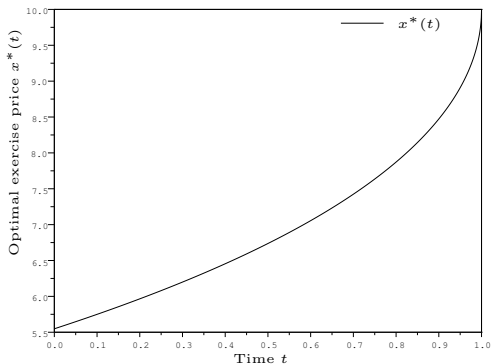
$$\dot{x}^*(t) = \frac{[\rho(\sigma^2 - 2r + q) - q][K - x^*(t)]^2 - 2K(r - q)[K - x^*(t)] + \sigma^2 K^2}{2\rho[K - x^*(t)]}$$

Solvable ODE via separation of variables

Valuation of American put (cont'd)

Numerical simulation

$$r = 0.03, \quad q = 0.01, \quad \sigma = 0.25, \quad T = 1, \quad K = 10, \quad \rho = 0.25$$

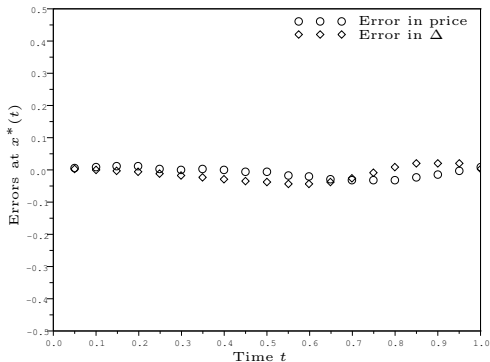


Valuation of American put (cont'd)

Note: PDE and final condition satisfied exactly for any $x^*(t)$, i.e., zero error

$$\text{Error in price} = p_a(x^*(t), t) - [K - x^*(t)]$$

$$\text{Error in } \Delta = \frac{\partial p_a}{\partial x}(x^*(t), t) - (-1)$$



Valuation of American call via a quasi put-call parity

Call payoff unbounded so Mellin transform not applicable

Let $p_a(x, t)$ solve

$$(\mathcal{L}p_a)(x, t) = [q(t)x - Kr(t)]H(y^*(t) - x) \quad (x \neq y^*(t), 0 \leq t < T)$$

Solution

$$p_a(x, t) = p_e(x, t) + K \int_t^T r(u) e^{-\int_t^u r(\tau) d\tau} N\left(-z_2\left(\frac{x}{y^*(u)}, t, u\right)\right) du \\ - x \int_t^T q(u) e^{-\int_t^u q(\tau) d\tau} N\left(-z_1\left(\frac{x}{y^*(u)}, t, u\right)\right) du$$

Note: $y^*(t)$ still unspecified

Valuation of American call via a quasi put-call parity (cont'd)

Define

$$c_a(x, t) = p_a(x, t) + x - K$$

Alternative expression (after lots of algebra!)

$$c_a(x, t) = c_e(x, t) + x \int_t^T q(u) e^{-\int_t^u q(\tau) d\tau} N\left(z_1\left(\frac{x}{y^*(u)}, t, u\right)\right) du \\ - K \int_t^T r(u) e^{-\int_t^u r(\tau) d\tau} N\left(z_2\left(\frac{x}{y^*(u)}, t, u\right)\right) du$$

If $q \equiv 0$, then

$$c_a(x, t) < c_e(x, t) \quad (0 \leq t < T),$$

i.e., early exercise never optimal for American call for non-dividend-paying assets

Valuation of American call via a quasi put-call parity (cont'd)

Verify

(i)

$$(\mathcal{L}c_a)(x, t) = [Kr(t) - q(t)x]H(x - y^*(t)) \quad (x \neq y^*(t), 0 \leq t < T)$$

(ii)

$$c_a(x, T) = p_a(x, T) + x - K = \max(x - K, 0)$$

(iii) Determine $y^*(t)$ such that

$$c_a(y^*(t), t) = y^*(t) - K, \quad \frac{\partial c_a}{\partial x}(y^*(t), t) = 1 \quad (0 \leq t < T)$$

Similar analysis to obtain integral equation satisfying both BCs and approximating ODE

Parity relations

Put-call parity for European options

$$c_e(x, t) - p_e(x, t) = xe^{-\int_t^T q(\tau) d\tau} - Ke^{-\int_t^T r(\tau) d\tau}$$

Quasi put-call parity for American options

$$c_a(x, t; y^*(t)) - p_a(x, t; y^*(t)) = x - K$$

Inequalities for American options

$$xe^{-\int_t^T q(\tau) d\tau} - K \leq c_a(x, t; y^*(t)) - p_a(x, t; y^*(t)) = x - K \leq x - Ke^{-\int_t^T r(\tau) d\tau}$$

Conclusion

- ▶ Reviewed the Mellin transform approach to European options with time-varying parameters
- ▶ Gave a unified approach to American options with time-varying parameters using Mellin transforms
- ▶ Presented new integral equations incorporating both smooth pasting conditions
- ▶ Derived solvable approximating ODEs for the optimal exercise prices
- ▶ Determined a quasi put-call parity for American options and used it to price the call from the put