Lunch at The Lab:

Volatility and Variance Swap Valuation in the COGARCH(1,1) Model

Matthew Couch

The University of Calgary

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Variance Swaps

A variance swap is a forward contract on the annualized variance of log asset returns. In practice, variance swaps are written on the realized variance, $\sigma_R^2(S)$ of daily closing prices.

Variance Swaps

At expiry, the holder of the variance swap receives receives some notional amount N (in dollars per annualized volatility) for each point by which the realized variance $\sigma^2(S)$ has exceeded the variance delivery price K_{var}. The payoff at expiry for a variance swap is thus equal to

$$N(\sigma_R^2(S) - K_{var})$$

Variance Swaps

The value of a variance swap is the present value of expected payoff of the swap in the risk neutral world

$$P(\sigma^2) = E\{e^{-rT}N(\sigma_R^2(S) - K_{var})\} = Ne^{-rT}(E\sigma_R^2(S) - K_{var})$$

Since no money is usually exchanged when the contract is written, to be arbitrage free the present value of the expected payoff should be equal to zero, ie

$$e^{-rT}E[N(\sigma_R^2(S)-K_{var})]=0 \Rightarrow E\sigma_R^2(S)-K_{var}=0$$

Thus the fair strike price of the swap should be equal to the expected value of the the realized variance.

Volatility Swaps

A volatility swap is a forward contract on the annualized standard deviation of the log asset returns, denoted σ_R. At expiry, the holder of the volatility swap receives receives some notional amount N (in dollars per annualized volatility) for each point by which the stocks realized variance $\sigma(S)$ has exceeded the volatility delivery price K_{vol}. The payoff at expiry for a volatility swap is thus equal to

$$N(\sigma_R(S) - K_{vol})$$

Volatility Swaps

The value of a a volatility swap is the present value of expected payoff of the swap in the risk neutral world

$$P(\sigma) = E\{e^{-rT} N(\sigma_R(S) - K_{vol})\} = Ne^{-rT}(E\sigma_R(S) - K_{vol})$$

The following approximation (Brockhaus-Long (2000))is useful in cases where the volatility is given by a stochastic process:

$$E\sqrt{\sigma_R^2} \approx \sqrt{E\sigma_R^2} - \frac{Var[\sigma_R^2]}{8(E\sigma_R^2)^{3/2}}$$

Volatility Swaps

Again, since no money is usually exchanged when the contract is written, to be arbitrage free the present value of the expected payoff should be equal to zero, ie

$$e^{-rT}E[N(\sigma_R(S) - K_{vol})] = 0 \Rightarrow E\sigma_R(S) - K_{vol} = 0$$

Thus the fair strike price of the swap should be equal to the expected value of the the realized variance.

Heston's Stochastic Volatility Model

Pricing volatility and variance swaps requires a stochastic volatility model, for example, one popular approach is the Heston model:

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t^1$$

$$d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dW_t^2$$

where $(W_t^1)_{t\geq 0}$ and $(W_t^2)_{t\geq 0}$ are independent standard Brownian motions.

Heston's Stochastic Volatility Model

here we approximate the realized variance, $\sigma_R^2(S)$ with the following integral

$$\sigma_R^2(S) = \frac{1}{T}\int_0^T \sigma_u^2 du$$

Heston's Stochastic Volatility Model

In this case the variance swap value is given by

$$EN(\sigma_R^2(S) - K_{var}) = N(E\sigma_R^2(S) - K_{var}) =$$

and

$$E\sigma_R^2(S) = \frac{1}{T}\int_0^T E\sigma_u^2 du = e^{-kt}(\sigma_0^2 - \theta^2) + \theta^2$$

(see Swishchuk (2004) or Brockhaus and Long (2000) for the proof)

Heston's Stochastic Volatility Model

The Volatility swap price may be derived similarly using the approximation:

$$E\sqrt{\sigma_R^2} \approx \sqrt{E\sigma_R^2} - \frac{Var[\sigma_R^2]}{8(E\sigma_R^2)^{3/2}}$$

Lévy Processes

Formal Definition

A càdlàg, adapted, real valued stochastic process $L = \{L_t, t \geq 0\}$ with $L_0 = 0$ a.s. is called a Lévy process if the following are satisfied:

- L has *independent increments*, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t \leq T$
- L has *stationary increments*, i.e. for any $s, t \geq 0$ the distribution of $L_{t+s} - L_t$ does not depend on t
- L is *stochastically continuous*, i.e. for all $t > 0$ and $\epsilon > 0$:

$$\lim_{s \to t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0$$

Lévy Processes

The characteristic function of L_t describes the distribution of each independent increment is given by $\phi(u) = e^{t\,\eta(u)}$ ($t > 0$ and $u \in \mathbb{R}$), where $\eta(u)$ is the characteristic exponent of the process.

Lévy-Khintchine Formula

The characteristic exponent of L_t can be be expressed as

$$\eta(u) = \mathrm{i}\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\,\mathbf{1}_{\{|x|<1\}}\nu(dx)$$

Lévy processes are often represented by their Lévy triplet (γ, σ^2, ν)

Compound Poisson Process

$L_t = \sum_{k=1}^{N_t} Y_k$ where (N_t) is a Poisson process with jump rate $\lambda > 0$, and (Y_k),$k = 1, 2, 3, ...$ are i.i.d. random variables, independent of N. The Levy measure of L has the representation

$$\nu_L(dx) = \lambda F_Y(dx)$$

The Variance Gamma Process

The variance gamma process may be defined as a time-changed Brownian motion with drift, where the time change is done with a gamma process, i.e. a process $(T_t)_{t\geq 0}$ where each T_t is gamma distributed:

$$f_{T_t} = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} e^{-\beta x}$$

for $x > 0$.

The Variance Gamma Process

If we let $B(t)$ be a standard Brownian motion, $\theta \in \Re$, $\sigma > 0$ and $\alpha = \beta = \frac{1}{\tau} > 0$ then the variance gamma process can be defined as:

$$V_t := \theta T_t + \sigma B(T_t)$$

for $t \geq 0$

Variance Gamma Process

If the the Levy measure of Variance Gamma Process has the Lebesgue density

$$\nu_L(dx) = \begin{cases} C exp(Gx)|x|^{-1}dx & \text{if } x < 0 \\ C exp(-Mx)x^{-1}dx & \text{if } x > \end{cases}$$

The GARCH (1,1) Model

The GARCH (1,1) model was introduced by Bollerslev in 1986 and is defined by the following: $Y_i = \sigma_i \epsilon_i$ with

$$\sigma_i^2 = \omega_0 + \lambda Y_{i-1}^2 + \delta \sigma_{i-1}^2$$

for $i = 1, 2, 3, ...$ and $\epsilon_i \ N(0, 1)$.

Nelson's Continuous time GARCH model

D. Nelson (1990) developed the following continuous time GARCH model as diffusion approximation. This model has separate, independent Brownian motions driving the volatility and asset price.

$$dG_t = \sigma dB_t$$

$$d\sigma_t^2 = (\omega - \theta\sigma_t^2)dt + \lambda\sigma_t^2 dW_t$$

where $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are independent standard Brownian motions.

The Continuous time GARCH model of Kazmerchuk et al

Kazmerchuk, Y., Swishchuk, A. and Wu, J. (2002) developed the following form of continuous time GARCH(1,1) [by rewriting the GARCH(1,1) equations as an SDE, this model inherits all of the parameters of the GARCH (1,1) model and maintains the form of the GARCH(1,1) equations.

$$dS_t = rS_t dt + \sigma(t, S_t) dW_t^*$$

$$\frac{d\sigma^2(t, S_t)}{dt} = \lambda V + \frac{\alpha}{\tau} \ln^2\left(\frac{S_t}{S_{t-\tau}}\right) - (\alpha + \lambda)\sigma^2(t, S_t)$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion and $W_t^* = \int_0^t \lambda(s) ds + W_t$.

- Kluppelberg, C., A. Lindner, and R. Maller (2004) introduce the COGARCH(1,1) model
- Haug, S., Klppelberg, C., Lindner, A. and Zapp 2007 develop method of moments estimation for the COGARCH(1,1) model
- Jan Kallsen, Bernhard Vesenmayer (2009) show that any COGARCH process can be represented as the limit in law of a sequence of GARCH(1,1) processes

The COGARCH(1,1) Equations

The COGARCH(1,1) Equations as described in Haug et al 2007

The COGARCH(1,1) Equations

$$dG_t = \sigma_t dL_t$$

$$d\sigma^2_{t+} = (\beta - \eta\sigma^2_t)dt + \phi\sigma^2_t d[L, L]^d_t$$

where $[L, L]^d_t = \sum_{0 \leq s \leq t} (\Delta L_s)^2$

The GARCH (1,1) Model

The GARCH (1,1) model is defined as $Y_i = \sigma_i \epsilon_i$ with

$$\sigma_i^2 = \omega_0 + \lambda Y_{i-1}^2 + \delta \sigma_{i-1}^2$$

for $i = 1, 2, 3, ...$ and ϵ_i $N(0, 1)$

The COGARCH (1,1) Model

Iterating the volatility in the previous expression we obtain

$$\sigma_i^2 = \omega_0 \sum_{k=0}^{i-1} \prod_{j=k+1}^{i-1} (\delta + \lambda \epsilon_j^2) + \sigma_0^2 \prod_{j=0}^{i-1} (\delta + \lambda \epsilon_j^2)$$

$$= \omega_0 \int_{k=0}^{i-1} exp \left\{ \sum_{j=\lfloor u \rfloor +1}^{i-1} log(\delta + \lambda \epsilon_j^2) \right\} + \sigma_0^2 exp \left\{ \sum_{j=0}^{i-1} log(\delta + \lambda \epsilon_j^2) \right\}$$

The COGARCH (1,1) Model

$$= \left[\omega_0 \int_{k=0}^{i} exp\left\{ \eta(\lfloor u \rfloor + 1) - \sum_{j=0}^{\lfloor u \rfloor} log(1+\phi\epsilon_j^2) \right\} + \sigma_0^2 \right]$$

$$\times exp\left\{ -\eta i + \sum_{j=0}^{i-1} log(1+\psi\epsilon_j^2) \right\}$$

where $\eta = -log(\delta)$ and $\phi = \frac{\lambda}{\delta}$

The COGARCH (1,1) Model

Replacing the noise terms ϵ_i with the jumps of a Levy process $\Delta L_t = L_t - L_{t-}$ we obtain the following expression:

$$\sigma_t^2 = (\omega_0 \int_0^t e^{X_s} ds + \sigma_0^2) e^{-X_{t-}}$$

$$X_t = -t\eta - \sum_{0<s\leq t} \log(1 + \phi(\Delta L_s)^2)$$

The COGARCH(1,1) process is then defined to be $dG_t = \sigma_t dL_t$

The COGARCH(1,1) Equations:

In Theorem 2.2 in Brockwell et al. (2006) it is shown that the volatility process 2 can also be defined as the solution to the SDE

$$d\sigma_{t+}^2 = (\alpha\eta - \eta\sigma_t^2)dt + \phi\sigma_t^2 d[L,L]_t^d$$

where $[L,L]_t^d = \sum_{0\leq s\leq t}(\Delta L_s)^2$

The Second Order Properties of the Log returns process G_t

Let $G_r^{(t)} = G_r - G_{r-t}$ and $\Psi(s) = -\eta s + \int_R ((1+\phi x^2)^s - 1)\nu_L(dx)$. Suppose that the Levy process $(L_t)t = 0$ has finite variance and zero mean, and that $\Psi(1) < 0$.

Then we have the following:

- $EG_r^{(t)} = 0$
- $E(G_r^{(t)})^2 = \frac{\beta t}{|\Psi(1)|} E(L_1^2)$
- $cov[(G_r^{(t)})^2, (G_r^{(s)})^2] = \frac{E(L_1^2 \beta^2}{|\Psi(1)|^3} (\frac{2}{\eta} + 2\sigma_L^2 - E(L - 1^2)) \times (\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|} (1 - e^{-r|\Psi(1)|})(e^{r|\Psi(1)|} - 1) e^{(s-t)|\Psi(1)|})$

Compound Poisson COGARCH(1,1) Process

$L_t = \sum_{k=1}^{N_t} Y_k$ where where (N_t) is a Poisson process with jump rate $\lambda > 0$, and (Y_k),$k = 1, 2, 3, ...$ are i.i.d. random variables, independent of N. The Levy measure of L has the representation

$$\nu_L(dx) = \lambda F_Y(dx)$$

Simulated Path: Compound Poisson COGARCH(1,1) Process

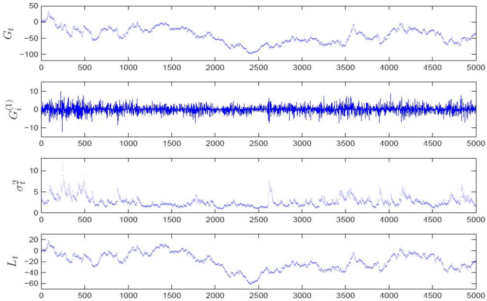

Figure: Compound Poisson COGARCH(1,1) Process (Haug 2006)

Variance Gamma COGARCH(1,1) Process

If we assume L is Variance Gamma with $E(L) = 0$ and $Var[L] = 1$, then the Levy measure of L has the Lebesgue density

$$\nu_L(dx) = \frac{C}{|x|} exp(-(2C)^{1/2}|x|)dx$$

for $x \neq 0$

Simulated path: Variance Gamma COGARCH(1,1) Process

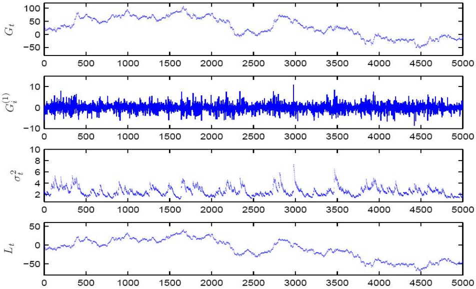

Figure: Variance Gamma COGARCH(1,1) Process (Haug 2006)

Asset Price Model

Let S_t be the asset price at time t, We assume that the log returns of stock prices follow the G_t, that is $S_t = e^{G_t}$

Pricing Variance Swaps Under the COGARCH(1,1) Model

Recall that the value of a variance swap is given by

$$P(\sigma^2) = Ne^{-rT}(E\sigma_R^2(S) - K_{var})$$

and $E\sigma_R^2(S)$ is the expected variance of the log asset returns

Pricing Variance Swaps Under the COGARCH(1,1) Model

Then the expected variance of the log returns of our asset is given by

$$E\sigma_R^2(S) = Var(\log(\frac{S_t}{S_0})) = Var(G_t - G_0) = E(G_t - G_0)^2$$

and we have

$$E\sigma_R^2(S) = \frac{\beta t}{|\Psi(1)|} E(L_1^2)$$

Pricing Variance Swaps Under the COGARCH(1,1) Model

Substituting we obtain the price variance swap for zero mean Levy Processes

Variance Swap Price

$$P(\sigma^2) = Ne^{-rT}(\frac{\beta T}{|\Psi(1)|}E(L_1^2) - K_{var})$$

Where $\Psi(s) = -\eta s + \int_R((1+\phi x^2)^s - 1)\nu_L(dx)$

Compound Poisson COGARCH(1,1) Process

In the case of the Compound Poisson COGARCH(1,1) Process we have

$$\Psi(s) = -\eta s + \lambda \int_R ((1+\phi y^2)^s - 1) F_Y(dy)$$

$$\Psi(1) = -\eta + \lambda \phi E Y^2$$

Thus the variance swap price is

$$P(\sigma^2) = N e^{-rT} \left(\frac{\beta T}{|-\eta + \lambda \phi E Y^2|} E(L_1^2) - K_{var} \right)$$

Compound Poisson COGARCH(1,1) Process

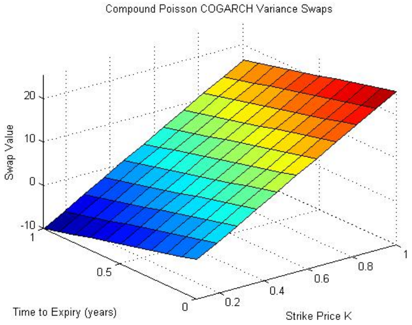

Figure: Compound Poisson COGARCH(1,1) Variance Swap Prices

Variance Gamma COGARCH(1,1) Process

Variance Gamma COGARCH(1,1) Process

$$\nu_L(dx) = \frac{C}{|x|} exp{-}(2C)^{1/2}|x|dx$$

$$\Psi(1) = -\eta + \phi$$

Thus the variance swap price is

$$P(\sigma^2) = Ne^{-rT}(\frac{\beta T}{|-\eta + \phi|}E(L_1^2) - K_{var})$$

Variance Gamma COGARCH(1,1) Process

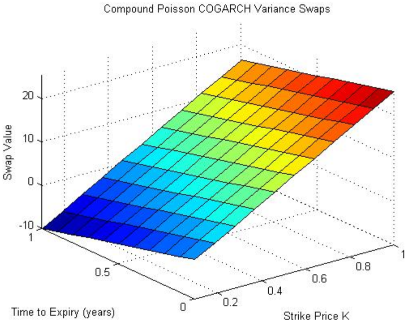

Figure: Variance Gamma COGARCH(1,1) Variance Swap Prices

Pricing Volatility Swaps Under the COGARCH(1,1) Process

Recall that the price of a variance swap is given by

$$P(\sigma) = Ne^{-rT}(E\sigma_R(S) - K_{vol})$$

where r is the risk-free discount rate corresponding to the expiration date T. We use the Brockhaus-Long (2000) approximation

$$E\sqrt{\sigma_R^2} \approx \sqrt{E\sigma_R^2} - \frac{Var[\sigma_R^2]}{8(E\sigma_R^2)^{3/2}}$$

We may apply the above approximation to the second order properties of the log returns process in order to determine the value of volatility swaps.

Compound Poisson COGARCH(1,1) Volatility Swap Prices

For example, the Compound Poisson Process we have the following:

$$\Psi(2) = -2\eta + 2\phi\lambda E[Y^2] + \phi^2\lambda E[Y^4]$$

We may thus calculate the value of the volatility swap using the rather long expression that results from substituting the above into the volatility swap price formula

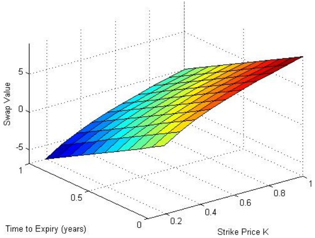

Figure:

Compound Poisson COGARCH(1,1) Volatility Swap Prices, $\phi = 0.038$, $\eta = 0.053$ and $beta = 0.04$ using the Brockhaus-Long approximation (left). Prices without the convexity correction (right)

Convexity Adjusted v.s. Naive Strike Fair Strike Prices

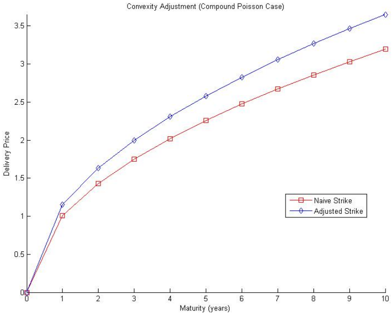

Figure: Convexity Adjustment

Variance Gamma COGARCH(1,1) Volatility Swap Prices

Similarly in the Variance Gamma COGARCH(1,1) Process

$$\Psi(2) = -2\eta + 2\phi 3\phi^2 C^{-1}$$

and may thus calculate the value of the volatility swap using the rather long expression that results from substituting the above into the volatility swap price formula

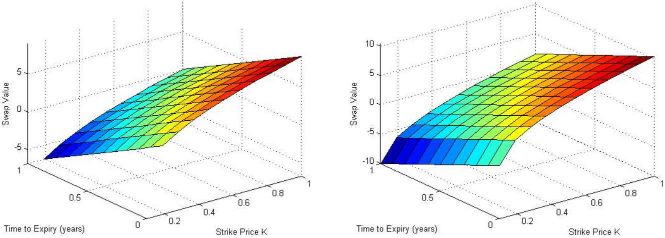

Figure:

Variance Gamma COGARCH(1,1) Volatility Swap Prices $C = 1$, $\phi = 0.038$, $\eta = 0.053$ and $beta = 0.04$ using the Brockhaus-Long approximation (left). Prices without the convexity correction (right)

Further work

- Calibration of the COGARCH model
- Comparison of swap values with other models, for example, discrete time GARCH and the Heston model

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