Applications of the method of maximum entropy in the mean to finance

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Overview:

1. Determination of risk pricing measures from market prices of risk
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    - Different families of risk measures.
    - Method of maximum entropy in the mean.
    - Numerical examples.
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1. Introduction

The problem of pricing actuarial and financial risk has received a great deal of attention in recent years.

Use of actuarial risk measures: for financial derivatives pricing (Goovaerts and Laeven (2008)) and for the relationship between risk measurement and decision making (Goovaerts, Kaas and Laeven (2005) and Goovaerts, Kaas and Laeven (2008))

In practice, there exist many different risk measures. There is no consensus about which is the best risk measure.

It is important to know which is the measure used for each firm.

Measurement of market risk:

Value at Risk: $VaR_\alpha(X) = \sup\{x : P[X > x] > 1 - \alpha\} = \inf\{x : P(X \leq x) \geq \alpha\}$.

VaR answers the question: What is the minimum loss incurred in the $(1-\alpha)\%$ worse cases of the position?

Definition: For a risk X and a confidence level $0 < \alpha < 1$, the $\alpha \times 100\%$ CVaR is:

$$CVaR_\alpha(X) = E_P[X \mid X \geq VaR_\alpha(X)].$$

CVaR answers the question: What is the expected loss incurred in the $(1-\alpha)\%$ worse cases of the position?

Families of market risk:

- Artzner, Delbaen, Eber and Heath (1999) proposed several properties that a risk measures should satisfy: notion of coherent risk measure.
- Follmer and Schied (2002) suggested that market risk may increase non–linearly with the value of the position, relaxing the positive homogeneity and sub–additivity conditions to convexity.
- Spectral measures were defined by Acerbi (2002) and studied by Kusuoka (2001): coherent risk measures that have two additional properties, law invariance and comonotone additivity
- Rockafellar et al. (2006) add two different sets of risk measures; deviation and bounded–in–expectation measures. These measures are not coherent.
- A distortion risk measure is the expectation of a new variable, with changed probabilities, re–weighting the initial distribution.

Different families of risk measures..

We consider a one period market model $(\Omega, \mathcal{F}, P)$. $\{S_j \,|\, j = 0, ..., N\}$ are the basic liabilities traded in the market. The final position is $X \in \mathcal{L}_2(P)$.

Definition: A risk measure ρ is coherent iff

1. Sub-additivity: For any $X, Y \in \mathbb{X}$, then $\rho(X + Y) \leq \rho(X) + \rho(Y)$,

2. Positive homogeneity: For any $X \in \mathbb{X}$ and $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$.

3. Translation invariance: For a fixed $X \in \mathbb{X}$ and any $a \in \mathbb{R}$, then $\rho(X+a) = \rho(X)+a$.

4. Monotonicity: Let $X, Y \in \mathbb{X}$ be such that $X \leq Y$ then $\rho(X) \leq \rho(Y)$.

Definition An element $\phi \in \mathcal{L}_1([0,1])$ (the class of Lebesgue integrable functions) is called an admissible risk spectrum if is increasing and $\phi \geq 0$ with $\|\phi\| = \int_0^1 \phi(t)dt = 1$.

Definition Let $\phi \in \mathcal{L}_1([0,1])$ be an admissible risk spectrum. The risk pricing measure

$$\rho_\phi(X) = \int_0^1 q_X(u)\phi(u)du$$

is called the spectral risk measure generated by ϕ.

Spectral measures were defined by Acerbi (2002) and studied by Kusuoka (2001): coherent risk measures have two additional properties, law invariance and comonotone additivity

Law invariance is important in practice as it is required for the estimation of the risk measure from empirical data.

Definition: a distortion function $g : [0,1] \longrightarrow [0,1]$ is non–decreasing and such that $g(0) = 0$ and $g(1) = 1$.

g is a distortion operator. It transforms S_X into a new survival function $g \circ S_X$.

It can be shown that the Choquet integral of a loss X

$$H_g[X] = \int_0^\infty g[S_X(x)]dx - \int_0^\infty \{1 - g[S_X(-x)]\}dx,$$

is equal to the expectation of X under the distorted distribution.

Theorem Let g a concave distortion function, and let H_g be the associated distorted risk pricing measure. Then $\phi(u) = g'(1-u)$ defines a spectral measure ρ_ϕ such that $\rho_\phi(X) = H_g(X)$.

VaR can be expressed as the Choquet integral with respect to the following distortion function:

$$g(y) = \begin{cases} 0 & \text{if } y < 1-\alpha \\ 1 & \text{if } y \geq 1-\alpha \end{cases}. \qquad (1)$$

$$VaR_\alpha(X) = \int_0^\infty g[S_X(y)]dy - \int_0^\infty \{1 - g[1 - S_X(-y)]\}dy = x_\alpha \ ,$$

where x_α is the percentile of the distribution of X

VaR uses only information on the loss frequency, not its severity. For instance, doubling the loss severity has little influence on VaR.

Note that here g is non–decreasing, with $g(0) = 0$ and $g(1) = 1$, piece–wise constant and non–concave. Hence the resulting distortion risk measure, VaR, is not coherent.

CVaR can also be expressed as a Choquet integral, with respect to the following distortion function:

$$g(y) = \begin{cases} \frac{y}{1-\alpha} & \text{if } y \leq 1-\alpha \\ 1 & \text{if } y \geq 1-\alpha \end{cases}. \quad (2)$$

Here g is a non–decreasing, continuous and concave distortion function.

The WT measure (Wang, 2002) used to price financial derivatives

$$g_{\alpha}(u) = \Phi[\Phi^{-1}(u) + v], \quad u \in (0,1)$$

The proportional hazard distortion functions are a special subclass of coherent distortion functions that relate nicely to spectral risk measures.

$$g_{PH}(u) = u^{\frac{1}{\gamma}}, \quad \gamma \geq 1$$

$$g_{DP}(u) = 1 - (1 - u^{\nu}), \quad \nu \geq 1$$

Objective

A new participant in the insurance services business wants to know how his competitors price risk.

$$\Downarrow$$

We want to devise a way to price other risks that is consistent with the prices of the already priced risks.

$$\Downarrow$$

We provide a nonparametric method for the construction of distortion functions from the observed prices of risk.

Our method consists of an application of the method of maximum entropy in the mean to obtain the distortion function.

Problem statement

Given

$$\begin{aligned} E^*[X] &= \int_0^\infty x dF_X^*(x) = \int_0^\infty g(1-F(x))dx = H_g(X) \\ &= \int_0^1 q_X(u) g'(1-u) du = \rho_\phi(X) \end{aligned}$$

We state our basic problem: given the market price π_i of a finite collection of risk positions X_i for $i = 1, ..., M$, find a spectral risk aversion function ϕ such that

$$\pi_i = \rho_\phi(X_i) = \int_0^1 q_{X_i}(u)\phi(u)du, \quad i = 1...., M$$

where to accommodate the condition $\int_0^1 \phi(u)du = 1$ we choose X_M such that $q_{X_M}(u) = 1$ and $\pi_M = 1$.

The first step consists of a discretization procedure $B = \frac{q_{X_i}(u_j)}{N}, \;\; u_j = \frac{j}{N}$:

$$\mathbf{B}\phi = \pi; \quad \phi \in \mathbf{K_o}.$$

where $\mathbf{K_o} = \{(\phi_1, ..., \phi_N) \,|\, \phi_1 < ... < \phi_j < \phi_{j+1} < ... < \phi_N\}$.

To simplify the description of the constraints, we set $\phi_1 = \psi_1$, $\phi_2 = \psi_1 + \psi_2$, ... and $\phi_N = \psi_N + ... + \psi_1$, or $\phi = \mathbf{C}\psi$ where $\mathbf{C}$ is the obvious lower diagonal matrix describing the change of coordinates. Setting $\mathbf{A} = \mathbf{BC} \Rightarrow \mathbf{A}\psi = \pi; \quad \psi \in \mathbf{K}$.

Clearly, once the vector ψ is at hand, the ϕ is easily recovered.

We solve the Fredholm equations using maximum entropy in the mean (Gamboa and Gzyl (1997)). This is a technique for transforming an ill-posed linear problem with a convex constraint into a simpler but non-linear optimization problem.

Basic methodology

To continue we need to select a reference measure $dQ^o(\xi)$ on $(\Omega, \mathcal{F})$.The only restriction is that the closure of the convex hull of $supp(Q)$ should be $\mathbf{K}$.

We define the class

$$\mathbb{P} = \{Q \,|\, Q << Q^o;\ AE_Q[\Psi] = \pi\}.$$

The explicit procedure to produce such Q's is known as the method of maximum entropy. By the rule

$$S^o_Q(Q) = -\int_\Omega \ln(\frac{dQ}{dQ^o})dQ$$

To guess the form of the density of the measure Q^* that maximizes S^o_Q, we consider the class of exponential measures on Ω

$$dQ_\lambda = \frac{e^{-<\lambda, \mathbf{A}\Psi>}}{Z(\lambda)} dQ^o$$

where the normalization factor is

$$Z(\lambda) = E^o_Q[e^{-<\lambda, \mathbf{A}\Phi>}].$$

We define the dual entropy function

$$\Sigma(\lambda) = \ln Z(\lambda) + < \lambda, \pi >$$

and the solution λ^* solve our problem.

The bounded case

Let us assume that for appropriate a and b, we know that $a \leq \psi_j \leq b \forall j$.

Also, since any point in $[a, b]$ is a convex combination of the end points, a simple assumption consists of putting

$$dQ^o(\xi) = \prod_{j=1}^{N} (p\delta_a(d\xi_j) + q\delta_b(d\xi_j)).$$

where $p + q = 1$. A similar assumption would consist of choosing a uniform distribution on $[a, b]$.

We consider also the unbounded case.

The next step consists of computing the normalization factor $Z(\lambda)$.

$$Z(\lambda) = \prod_{j=1}^{N} \zeta((\mathbf{A}^*\lambda)_j)),$$

where $\zeta(\tau)$ is the Laplace transform of $p\delta_a(dx) + q\delta_b(dx)$, that is

$$\zeta(\tau) = \int e^{-x\tau} p\delta_a(dx) + q\delta_b(dx) = pe^{-a\tau} + qe^{-b\tau}.$$

We calculate λ^*, solution of the dual problem. It is easy to see that the maxentropic reconstruction ψ^* is given by

$$\psi_j^* = ap_j^* + bq_j^*$$

$$p_j^* = \left(\frac{e^{-a(A^*\lambda^*)_j}}{e^{-a(A^*\lambda^*)_j} + e^{-b(A^*\lambda^*)_j}}\right), \quad q_j^* = 1 - p_j^*.$$

Now the ϕ_j must be recovered from the ψ_j

Numerical examples

Simplest situation consisting of assuming that we are presented with the risk price of a liability which we know to have been priced coherently, but using a distortion function unknown to us.

We shall consider a risk known to be distributed according to either a $U(0,1)$, a $Pareto(0,2)$, a $Gamma(1,2)$ or a $Beta(2,4)$ distribution.

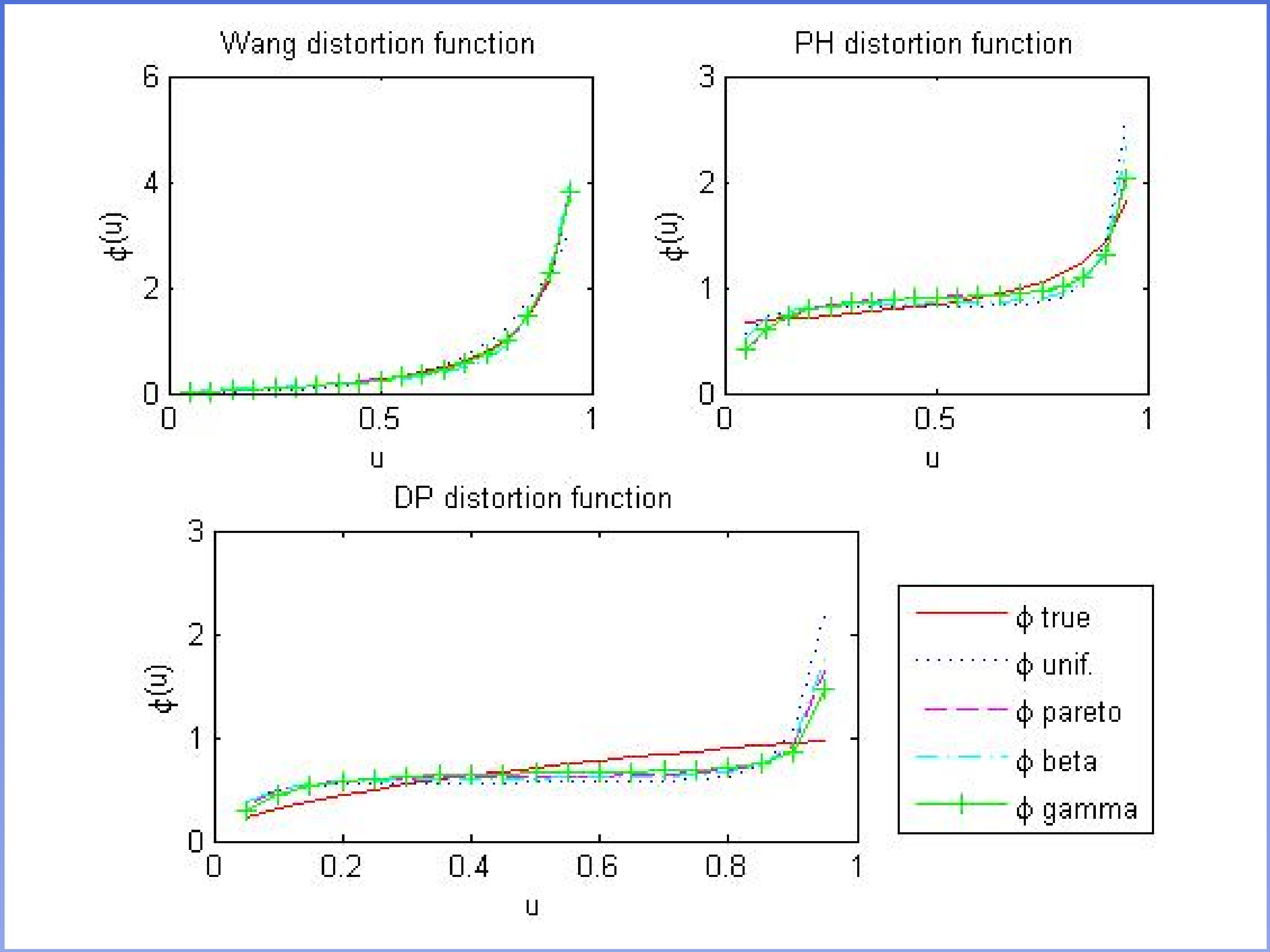
The computation of the risk price π of each liability was carried out with a Dual Power, PH and Wang distortion function. The parameters we use throughout are 1.5 for the proportional hazard and the dual power distortion function, and 0.05 for the Wang distortion function.

We indicate the reconstruction errors computed as $|\pi_k - (A\phi)^*_k|$, where ϕ^* is found as described. For this we considered $p = q = 1/2$ and $a = 0; b = 6$.

Table 1: Reconstruction errors

Distortion	*Wang*		*PH*		*DP*	
	π	*error*	π	*error*	π	*error*
Uniform	0.51	0.577×10^{-10}	0.58	0.46×10^{-12}	0.4	0.29×10^{-10}
Pareto	3.96	0.91×10^{-10}	4.03	0.46×10^{-12}	3.0	0.09×10^{-10}
Gamma	2.69	0.81×10^{-11}	2.2	0.29×10^{-10}	1.55	0.14×10^{-11}
Beta	0.34	0.96×10^{-11}	0.34	0.43×10^{-10}	0.26	0.79×10^{-10}

Reconstructed ϕ^* from prices



The important thing is not that the reconstructed ϕ^*'s look like the true one, but that the reconstructed error is small. These ϕ^*'s can then be used to price other risks.

Table 2: Error in risk price estimated with reconstructed ϕ^*.

Distortion	$Wang$		PH		DP	
	π^*	$error$	π^*	$error$	π^*	$error$
$Uniform$	0.94	0.00039	1.02	0.0033	0.76	0.001
$Pareto$	0.95	0.003	1.03	0.00019	0.74	0.002
$Gamma$	0.98	0.002	1.05	0.002	0.72	0.003
$Beta$	0.96	0.003	1.03	0.004	0.77	0.002

We consider a new $U(0,2)$ liability and compute its risk price according to the same three spectral risk functions as above, we compare it with the risk price computed with the ϕ^* computed using the reconstructed spectral risk functions obtained above.

We display the reconstruction error of each risk when the market prices of 4 liabilities are used to reconstruct one single spectral risk function.

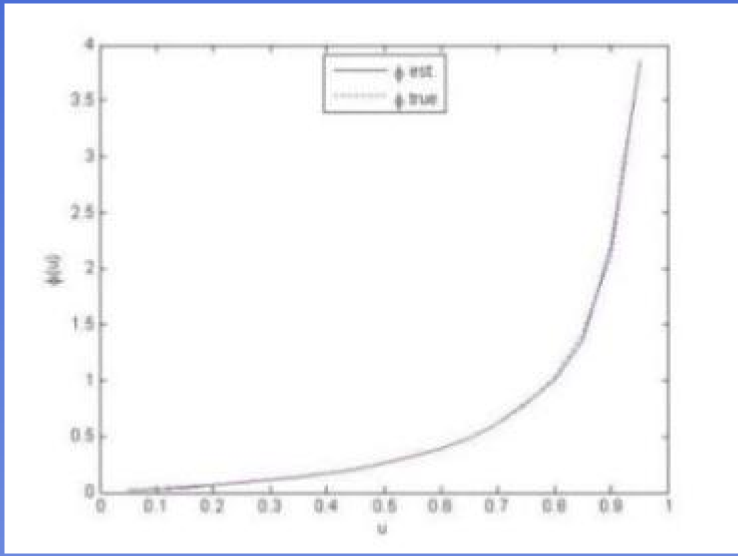
This time we considered a $U(0,1)$, a $Pareto(0,2)$, a $Gamma(2,4)$ and a $Beta(1,2)$, and the 4 liabilities were simultaneously priced with each risk aversion function.

Table 3: Error of reconstruction risk price

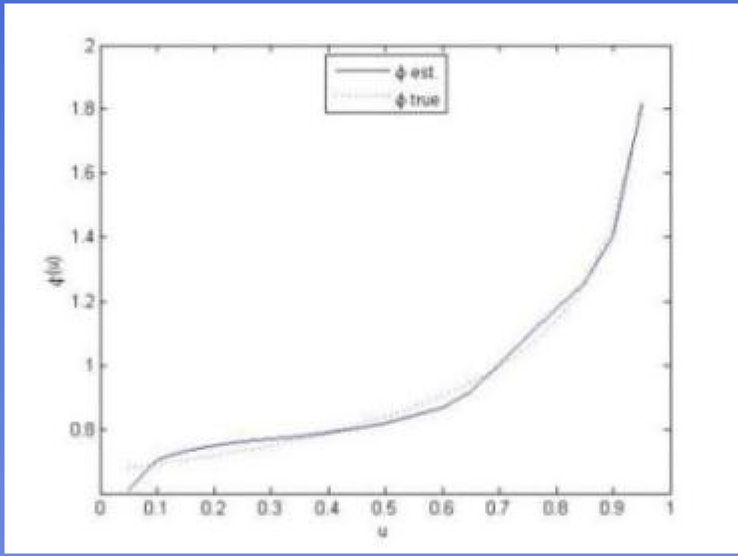
$Wang$		PH		DP	
π	$error$	π	$error$	π	$error$
0.52	0.12×10^{-8}	0.55	0.15×10^{-5}	0.39	0.41×10^{-6}
4.01	0.001×10^{-8}	4.15	0.07×10^{-5}	2.79	0.012×10^{-6}
0.32	0.44×10^{-8}	0.35	0.34×10^{-5}	0.25	0.88×10^{-6}
2.53	0.02×10^{-8}	2.00	0.044×10^{-5}	1.60	0.07×10^{-6}

The method recognizes the distortion function regardless of the liability.

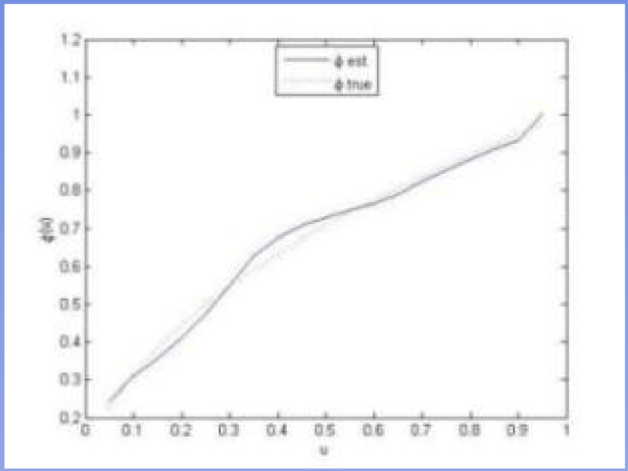
Wang distortion



Proportional Hazard



Dual Power



2. Determination of the risk neutral distribution from market option prices

An interesting inverse problem in finance is that of determining a risk neutral measure from market data.

The risk neutral measure is necessary for determining the fair prices of new derivatives having a given asset as underlying.

The mathematical problem:

$$p_i = e^{-rT} \int_0^\infty f_i(s)\phi(s)\rho(s)ds; \quad 0 = 1, ..., N.$$

where $\rho(s)$ is the physical law for $S(T)$, and r is the spot rate for the period.

The solution can be interpreted as providing a density $\phi(s)$ with respect to a physical law $\rho(s)$ which makes the market prices consistent among themselves.

We can rewrite the problem as:

$$\begin{aligned} 1 &= \int_0^\infty m(s)ds \\ e^{rT}p_i &= \int_0^\infty f_i(s)m(s)ds; \text{ for i} = 0, ..., \text{N}, \end{aligned} \qquad (3)$$

where we take $m(s) = \phi(s) * \rho(s)$, $f_0(s) = s$ and the cases $i = 1, ...N$ describe the cash flow of derivatives of European type.

We apply the method of maximum entropy in the mean.

The information available may or may not contain the present value of the asset as data.

Numerical Implementation

Reconstruction of the RNM (risk neutral measure) when the prices observed are risk neutral prices in the BS world.

To determine the free parameter b we propose to find it by minimizing the difference between the price calculated with the reconstructed RNM and the true observed price with a different strike price.

1. The underlying is known. The observed prices are call and put prices with $K = 1.6$.

2. Similar to case 1 but the underlying is known.

3. No information is available about price of the underlying, but we know the prices of a call with strike $K = 1.7$ and a put with strike $K = 1.4$.

4. Assume this time that we know the price of the underlying and we also know the price of a call with strike $K = 1.7$ and a put with strike $K = 1.4$.

We begin by comparing the prices computed with the reconstructed RNM against the true risk neutral prices. The errors are listed as %

Table 4: Reconstruction errors %

	Case 1		Case 2		Case 3		Case 4	
	Call	**Put**	**Call**	**Put**	**Call**	**Put**	**Call**	**Put**
1.4	0.009	0.036	0.068	0.334	0.223	0.013	0.005	0.014
1.5	0.053	0.095	0.018	0.055	0.381	0.022	0.017	0.018
1.6	0.119	0.045	0.637	0.296	0.863	0.082	0.693	0.300
1.7	1.112	0.153	0.006	0.009	0.036	0.179	0.034	0.007

The plots display the data, the true RNM and the reconstructed RNM in each of the four cases itemized above.

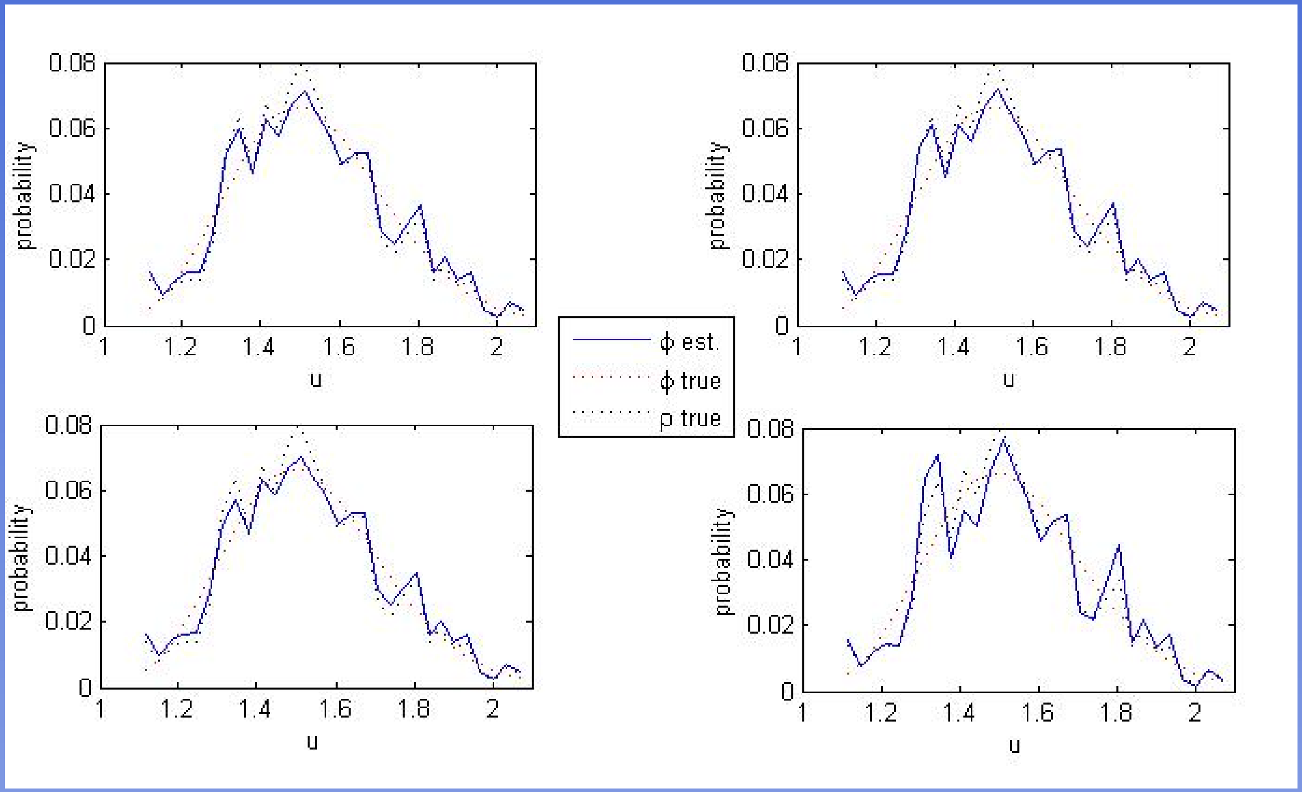


Figure 1: RNM's reconstructed from different data

A measure of misfit between the reconstructed and the true RNM's is computed by means of the true L_1-distance.

Table 4

Case	$norm$	b	$E(error)$
Case 1	0.037	2.3	0.0021
Case 2	0.035	2.4	0.0065
Case 3	0.031	2.6	0.0089
Case 4	0.039	1.5	0.0071

Here $E(error)$ is the the total reconstruction error of 8 options, (using expected value as the measure).

Some observations

To verify the robustness of the method, we carried out the procedure 500 times. The sample mean of the L_1-distance was 0.037 with a sample standar deviation equal to 0.0076.

We studied the influence of the value b.

We have solved the same problem when the prices include risk premia and/or transaction prices.

$$p_i = e^{-rT} \int_0^\infty f_i(s)\phi(s)\rho(s)ds + \epsilon_i; \quad 0 = 1, ..., N.$$