Smiling for the delayed Volatility Swap

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Overview

- Drift-adjusted Heston model incorporating delay for the volatility
- Closed formula for the Volatility Swap strike K_{vol}
  - Realized Variance $V = \frac{1}{T}\int_0^T V_s ds$
  - Brockhaus & Long approximation:

$$K_{vol} := \mathbb{E}(\sqrt{V}) \approx \sqrt{\mathbb{E}(V)} - \frac{var(V)}{8\mathbb{E}(V)^{\frac{3}{2}}}$$

- Application of the change of time method
- Calibration: semi-closed formula available for call options
- Numerical example: calibration on EURUSD, Sept. 30th 2011

Pricing Variance Swaps for Stochastic Volatilities with Delay (Swishchuk, 2005)

- First attempt for continuous GARCH model with delay:

$$\frac{dV_t}{dt} = \gamma\theta^2 + \frac{\alpha}{\tau}\ln^2\left(\frac{S(t)}{S(t-\tau)}\right) - (\alpha+\gamma)V_t$$

- We have by Ito's lemma:

$$\ln\frac{S(t)}{S(t-\tau)} = \int_{t-\tau}^{t}(\mu - \frac{1}{2}V_s)ds + \int_{t-\tau}^{t}\sqrt{V_s}dZ_s^{\mathbb{P}}$$

- Incorporated $\mathbb{P}-$drift of $\ln\frac{S(t)}{S(t-\tau)}$ in the square term:

$$\frac{dV_t}{dt} = \gamma\theta^2 + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t}\sqrt{V_s}dZ_s^{\mathbb{Q}} - (\mu - r)\tau\right]^2 - (\alpha+\gamma)V_t$$

Pricing Variance Swaps for Stochastic Volatilities with Delay (Swishchuk, 2005)

- They get the following delayed ODE for $v_t := \mathbb{E}^{\mathbb{Q}}(V_t)$

$$\frac{dv_t}{dt} = \gamma\theta^2 + \alpha\tau(\mu - r)^2 + \frac{\alpha}{\tau}\int_{t-\tau}^{t} v_s ds - (\alpha+\gamma)v_t$$

- Solution:
  - $v_t = X + (v_0 - X)e^{\rho t}$, $X = \theta^2 + \frac{\alpha\tau(\mu-r)^2}{\gamma}$
  - ρ is given by $\rho = -\alpha - \gamma + \frac{\alpha}{\rho\tau}(1 - e^{-\rho\tau})$
  - ρ is unique, $\rho < 0$, $\rho + \gamma \neq 0$
- VarSwap price:

$$K_{var} = \mathbb{E}^{\mathbb{Q}}(V) = \frac{1}{T}\int_0^T v_s ds = X + (v_0 - X)\frac{e^{\rho T} - 1}{\rho T}$$

Model

- Asset price: $dS_t = (r - q)S_t dt + S_t\sqrt{V_t} dZ_t^{\mathbb{Q}}$
- Notice previous GARCH-type model is similar to Heston model as $\lim_{\tau\to 0} \mathbb{E}^{\mathbb{Q}}(V_t^{GARCH}) = \mathbb{E}^{\mathbb{Q}}(V_t^{Heston})$ (because $\lim_{\tau\to 0} \rho = -\gamma$)
- "Moment matching" adjusted Heston model to account for the delay :

$$dV_t = \left[\gamma(\theta^2 - V_t) + \epsilon_\tau(t)\right] dt + \delta\sqrt{V_t} dW_t^{\mathbb{Q}}$$

- The adjustment is given by:

$$\begin{aligned} \epsilon_\tau(t) &= \alpha\tau(\mu - r)^2 + \frac{\alpha}{\tau}\int_{t-\tau}^{t} v_s ds - \alpha v_t \\ &= \alpha\tau(\mu - r)^2 + (v_0 - X)(\rho + \gamma)e^{\rho t} \end{aligned}$$

- We also have $\lim_{\tau\to 0} \sup_{t\in\mathbb{R}^+} |\epsilon_\tau(t)| = 0$

Change of time Method (Swishchuk, 2004)

- $x_t := -(v_0 - X)e^{(\gamma+\rho)t} + e^{\gamma t}(V_t - X)$
- By Ito's lemma:

$$\begin{aligned} dx_t &= \delta e^{\gamma t}\sqrt{(x_t + (v_0 - X)e^{(\gamma+\rho)t})e^{-\gamma t} + X}\, dW_t^{\mathbb{Q}} \\ &= f(t, x_t)dW_t^{\mathbb{Q}} \end{aligned}$$

- $x_t = \tilde{W}_{\phi_t}$, where $\tilde{W}_t$ is a $\mathcal{F}_{\phi_t^{-1}}$ adapted $\mathbb{Q}-$Brownian motion.

$$\begin{aligned} \phi_t &= \int_0^t f^2\left(s, \int_0^s f(u, x_u)dW_u^{\mathbb{Q}}\right) ds \\ \tilde{W}_t &= \int_0^{\phi_t^{-1}} f(s, x_s)dW_s^{\mathbb{Q}} \\ \phi_t^{-1} &= \int_0^t f^{-2}\left(\phi_s^{-1}, \int_0^{\phi_s^{-1}} f(u, x_u)dW_u^{\mathbb{Q}}\right) ds \end{aligned}$$

Change of time Method (Swishchuk, 2004)

- $V_t = X + (v_0 - X)e^{\rho t} + e^{-\gamma t}\tilde{W}_{\phi_t}$
- $\mathbb{E}^{\mathbb{Q}}(\tilde{W}_{\phi_t}) = 0$

$$\mathbb{E}^{\mathbb{Q}}(\tilde{W}^2_{\phi_t}) = \mathbb{E}^{\mathbb{Q}}(\phi_t) = \delta^2 \left[X \left(\frac{e^{2\gamma t} - 1}{2\gamma} \right) + (v_0 - X) \left(\frac{e^{(2\gamma+\rho)t} - 1}{2\gamma + \rho} \right) \right]$$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(\tilde{W}_{\phi_t}\tilde{W}_{\phi_s}) &= \mathbb{E}^{\mathbb{Q}}(\phi_t \wedge \phi_s) = \mathbb{E}^{\mathbb{Q}}(\phi_{t\wedge s}) \\ &= \delta^2 \left[X \left(\frac{e^{2\gamma(t\wedge s)} - 1}{2\gamma} \right) + (v_0 - X) \left(\frac{e^{(2\gamma+\rho)(t\wedge s)} - 1}{2\gamma + \rho} \right) \right] \end{aligned}$$

VolSwap Strike

- $K_{vol} = \sqrt{X + (v_0 - X)\frac{e^{\rho T}-1}{\rho T}} - \frac{Var(V)}{8\left(X+(v_0-X)\frac{e^{\rho T}-1}{\rho T}\right)^{\frac{3}{2}}}$

- Recall $V = \frac{1}{T}\int_0^T V_s ds$, therefore:

$$\begin{aligned}
Var(V) &= \frac{1}{T^2}\int_0^T\int_0^T \mathbb{E}^{\mathbb{Q}}(V_t V_s)dtds - \mathbb{E}^{\mathbb{Q}}(V)^2 \\
&= \frac{1}{T^2}\int_0^T\int_0^T e^{-\gamma(t+s)}\mathbb{E}^{\mathbb{Q}}(\tilde{W}_{\phi_t}\tilde{W}_{\phi_s})dtds \\
&= \frac{\delta^2 e^{-2\gamma T}}{2T^2\gamma^3}\Bigg[X\left(2\gamma T e^{2\gamma T} + 4e^{\gamma T} - 3e^{2\gamma T} - 1\right) + \frac{\gamma}{2\gamma+\rho}(v_0 - X)\Bigg(2e^{2\gamma T}\Bigg(-2\frac{\gamma}{\rho} \\
&\quad -1\Bigg) - 4\gamma e^{\gamma T}\left(\frac{e^{(\gamma+\rho)T}-1}{\gamma+\rho}\right) + 4e^{\gamma T}\left(1 + \frac{\gamma}{\rho}e^{(\gamma+\rho)T}\right) - 2\Bigg)\Bigg]
\end{aligned}$$

- Letting $\tau \to 0$, we get the result in (Swishchuk, 2004)

Calibration

- Heston model with time-dependent long-range variance $\tilde{\theta}^2(t) = X + (v_0 - X)\frac{(\rho+\gamma)}{\gamma} e^{\rho t}$

- Call price C_0

$$C_0 = e^{-rT}\left[\frac{1}{2}(F-K) + \frac{1}{\pi}\int_0^\infty (Fh_1(u) - Kh_2(u))du\right]$$

$$h_1(u) = \Re\left(\frac{e^{-iu\ln(K)}\varphi(u-i)}{iuF}\right), h_2(u) = \Re\left(\frac{e^{-iu\ln(K)}\varphi(u)}{iu}\right)$$

$$\varphi(u) = e^{C(T,u)+v_0 D(T,u)+iu\ln(F)}$$

- ODE for C and D:

$$\frac{dD(t,u)}{dt} - \frac{\delta^2}{2}D^2(t,u) + (\gamma - iuc\delta)D(t,u) + \frac{1}{2}(u^2 + iu) = 0$$

$$\frac{dC(t,u)}{dt} = \gamma\tilde{\theta}^2(t)D(t,u)$$

$$\Rightarrow C(t,u) = \gamma X\int_0^t D(s,u)ds + (v_0 - X)(\rho+\gamma)\int_0^t e^{\rho s}D(s,u)ds$$

Numerical Methods

- least-squares minimization procedure that we perform via MATLAB (function *lsqnonlin*)
- The Heston integral C_0 is computed via the MATLAB function *quadl* that uses a recursive adaptive Lobatto quadrature.
- The integral $\int_0^t e^{\rho s} D(s,u) ds$ is computed via a composite Simpson's rule with 100 points.

Numerical Results

- Data: September 30^{th} 2011 for underlying EURUSD (source: Bloomberg). The drift $\mu = 0.0188$ is estimated from 7.5Y of daily close prices (source: www.forexrate.co.uk).
- The calibrated parameters are $(v_0, \gamma, \theta^2, \delta, c, \alpha, \tau) = (0.0293, 2.2021, 0.0394, 0.5988, -0.5338, 0.0178, 0.0075)$ and we compute $\rho = -2.2019$

	ATM	25D Call	25D Put	10D Call	10D Put
1M	44	28	2	53	25
2M	8	12	3	2	18
3M	0	16	3	2	9
4M	14	33	5	19	15
6M	6	22	0	17	3
9M	9	3	6	1	6
1Y	29	19	8	9	3

Table 1: Absolute Calibration Error (in bp of the BS volatility)

Numerical Results

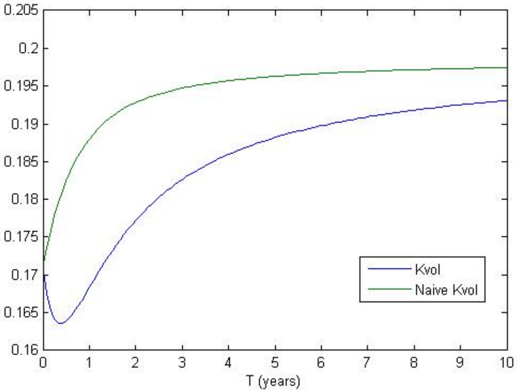

Figure 1: Naive Volatility Swap Strike Vs. Adjusted Volatility Swap Strike

Numerical Results

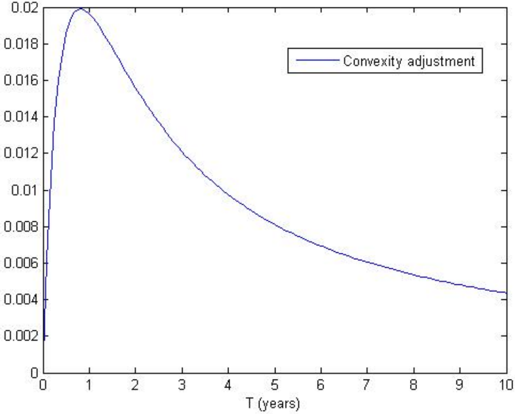

Figure 2: Convexity Adjustment

Main References

- Swishchuk (2005): *Modeling and pricing of variance swaps for stochastic volatilities with delay*
- Swishchuk (2004): *Modeling of Variance and Volatility Swaps for Financial markets with Stochastic volatilities*
- Mikhailov, Noegel (2003): *Heston's Stochastic Volatility Model Implementation, Calibration and Some Extensions*
- Kahl, Jäckel (2006): *Not-so-complex logarithms in the Heston model*