An Introduction to Lévy Processes with Applications in Finance

Antonis Papapantoleon

Presented by:
Kevin Malenfant

October 8, 2008

Introduction

Why?

- Describe the observed reality of financial markets in a more accurate way than models based on Brownian motion
  - Asset price processes have jumps or spikes
  - Empirical distribution of asset returns exhibits fat tails and skewness
  - In the ‘risk-neutral world, we observe that implied volatilities are constant neither cross strike nor across maturities

Lévy processes?

- Processes with independent and stationary increments are named *Lévy processes* after the French mathematician Paul Lévy (1886- 1971)

Lévy process

Let $(\Omega,\ \mathcal{F},\ \mathrm{F},\ P)$ be a filtered probability space, where $\mathcal{F} = \mathcal{F}_T$ and the filtration $\mathrm{F} = (\mathcal{F})_{t\in[0,T]}$ satisfies the usual conditions

Definition 2.1

A càdlàg, adapted, real valued stochastic process $L = (L_t)_{t\geq 0}$ with $L_0 = 0$ a.s. is called a *Lévy process* if the following conditions are satisfied:

(L1): L has *independent increments*, i.e. $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t \leq T$.

(L2): L has *stationary increments*, i.e. for any $s,\ t \geq 0$ the distribution of $L_{t+s} - L_t$ does not depend on t.

(L3): L is *stochastically continuous*, i.e for every $t \geq 0$ and $\epsilon > 0$:

$$\lim_{s\rightarrow t} P(|L_t - L_s| > \epsilon) = 0.$$

Infinitely divisible

Let X be a real valued random variable, denote its characteristic function by φ_X and its law by P_X, hence $\varphi_X(u) = \int_{\mathbb{R}} e^{iux} P_X(\mathrm{d}x)$.

Definition 4.1

The law P_X of a random variable X is **infinitely divisible**, if for all $n \in \mathbb{N}$ there exist i.i. $\mathrm{d}.$ random variables $X_1^{(1/n)}, \dots, X_n^{(1/n)}$ such that

$$X \overset{\mathrm{d}}{=} X_1^{(1/n)} + \ldots + X_n^{(1/n)}. \tag{4.1}$$

Infinitely divisible

Alternatively, we can characterize an infinitely divisible random variable using its characteristic function.

Definition 4.2

The law of a random variable X is **infinitely divisible**, if for all $n \in \mathbb{N}$, there exists a random variable $X^{(1/n)}$, such that

$$\varphi_X(u) = (\varphi_{X^{(1/n)}}(u))^n \tag{4.3}$$

Infinitely divisible

Example 4.3 (Normal distribution)

Using the second definition, we can easily see that the Normal distribution is infinitely divisible. Let $X \sim \mathrm{Normal}(\mu,\ \sigma^2)$, then we have

$$\begin{aligned}
\varphi_X(u) &= \exp[iu\mu - \frac{1}{2}u^2\sigma^2] \\
&= \exp\left[n\left(iu\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n}\right)\right] \\
&= \left(\exp\left[u\frac{\mu}{n} - \frac{1}{2}u^2\frac{\sigma^2}{n}\right]\right)^n \\
&= (\varphi_{X^{(1/n)}}(u))^n
\end{aligned}$$

where $X^{(1/n)} \sim \mathrm{Normal}\left(\frac{\mu}{n},\ \frac{\sigma^2}{n}\right)$.

Infinitely divisible

Example 4.4 (Poisson distribution)

Following the same procedure, we can easily deduce that the Poisson distribution is infinitely divisible. Let $X \sim \mathrm{Poisson}(\lambda)$, then we have

$$\begin{aligned}
\varphi_X(u) &= \exp[\lambda(\mathrm{e}^{iu} - 1)] \\
&= \exp\left[n\frac{\lambda}{n}(\mathrm{e}^{iu} - 1)\right] \\
&= \left(\exp\left[\frac{\lambda}{n}(\mathrm{e}^{iu} - 1)\right]\right)^n \\
&= \left(\varphi_{X^{(1/n)}}(u)\right)^n
\end{aligned}$$

where $X^{(1/n)} \sim \mathrm{Poisson}\left(\frac{\lambda}{n}\right)$.

Infinitely divisible

Examples

- compound Poisson distribution
- exponential
- Γ-distribution
- geometric
- negative binomial
- Cauchy distribution
- strictly stable distribution

Counter-examples

- uniform
- binomial

Lévy-Khintchine formula

Theorem 4.6

The law P_X of a random variable X is infinitely divisible if and only if there exists a triplet $(b,\ c,\ \nu)$, with $b \in \mathbb{R},\ c \in \mathbb{R}_+$ and a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty$, such that

$$\mathrm{E}[e^{iuX}] = \exp[ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}})\nu(\mathrm{d}x)]. \tag{4.4}$$

Lévy-Itô decomposition

Theorem 5.1

Consider a triplet $(b,\ c,\ \nu)$ where $b \in \mathbb{R},\ c \in \mathbb{R}_+$ and ν is a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty$. Then, there exists a probability space $(\Omega,\ \mathcal{F},\ P)$ on which four independent Lévy processes exist, $L^{(1)},\ L^{(2)},\ L^{(3)}$ and $L^{(4)}$, where $L^{(1)}$ is a constant drift, $L^{(2)}$ is a Brownian motion, $L^{(3)}$ is a compound Poisson process and $L^{(4)}$ is a square integrable (pure jump) martingale with an a.s. countable number of jumps on each finite time interval of magnitude less that 1. Taking $L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$, we have that there exists a probability space on which a Lévy process $L = (L_t)_{t \geq 0}$ with characteristic exponent

$$\psi(u) = iub - \frac{u^2 c}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}})\nu(\mathrm{d}x) \tag{5.1}$$

for all $u \in \mathbb{R}$, is defined.

Lévy-Khintchine formula (Outline of Proof)

Outline of Proof

We split the Lévy exponent (5.1) into four parts

$$\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \psi^{(4)}$$

where

$$\psi^{(1)}(u) = iub, \ \ \psi^{(2)}(u) = \frac{u^2 c}{2}$$

$$\psi^{(3)}(u) = \int_{|x|\geq 1} (e^{iux} - 1)\nu(\mathrm{d}x),$$

$$\psi^{(4)}(u) = \int_{|x|<1} (e^{iux} - 1 - iux)\nu(\mathrm{d}x).$$

Lévy-Khintchine formula (Outline of Proof)

Outline of Proof (Continued)

- let $\triangle L^{(4)}$ denote the jumps of the Lévy process $L^{(4)}$
- let $\mu^{L^{(4)}}$ denote the random measure counting the jumps of $L^{(4)}$
- construct a compensated compound Poisson process

$$L_t^{(4,\epsilon)} = \sum_{0 \le s \le t} \triangle L_s^{(4)} \mathbb{1}_{\{1 > |\triangle L_s^{(4)}| > \epsilon\}} - t\left(\int_{1>|x|>\epsilon} x\nu(\mathrm{d}x)\right)$$

$$= \int_0^t \int_{1>|x|>\epsilon} x\mu^{L^{(4)}}(\mathrm{d}x, \mathrm{d}s) - t\left(\int_{1>|x|>\epsilon} x\nu(\mathrm{d}x)\right)$$

Lévy-Khintchine formula (Outline of Proof)

Outline of Proof (Continued)

- Show that the jumps of $L^{(4)}$ form a Poisson point process.
- Get that the characteristic function of $L^{(4,\epsilon)}$ is

$$\psi^{(4,\epsilon)}(u) = \int_{\epsilon<|x|<1} (e^{iux} - 1 - iux)\nu(\mathrm{d}x).$$

- There exists a Lévy process $L^{(4)}$ which is a square integrable martingale and $L^{(4,\epsilon)} \longrightarrow L^{(4)}$ uniformly on $[0,\ T]$ as $\epsilon \longrightarrow 0+$ with Lévy exponent $\psi^{(4)}$.
- Therefore, we can decompose any Lévy process into four independent Lévy processes $L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$, i.e.

$$L_t = bt + \sqrt{c}W_t + \int_0^t \int_{|x|\geq 1} x\mu^L(\mathrm{d}s,\ \mathrm{d}x) \tag{5.2}$$

$$+(\int_0^t \int_{|x|<1} x\mu^L(\mathrm{d}s,\ \mathrm{d}x) - t\int_{|x|<1} x\nu(\mathrm{d}x))$$

The Lévy measure

Lévy measure

- The Lévy measure ν is a measure on $\mathbb{R}$ that satisfies

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} (1 \wedge |x|^2) u(\mathrm{d}x) < \infty. \tag{6.1}$$

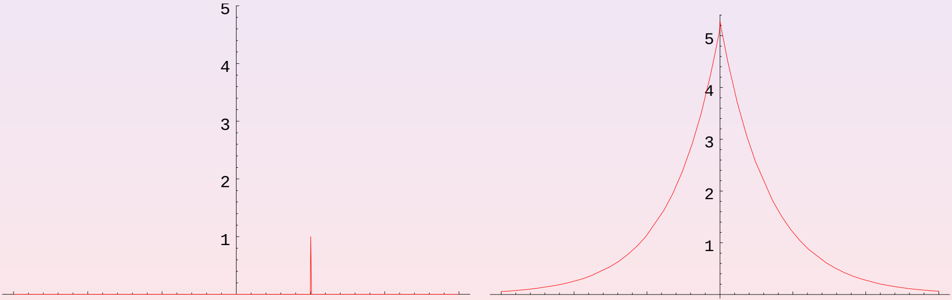
- A Lévy measure has no mass at the origin, but singularities (i.e. infinitely many jumps) can occur around the origin (i.e. small jumps).
- Intuitively speaking, the Lévy measure describes *the expected number of jumps of a certain height in a time interval of length 1*.

Example

The Lévy measure of the Lévy jump-diffusion is $\nu(\mathrm{d}x) = \lambda \cdot F(\mathrm{d}x)$; from that we can deduce that the expected number of jumps, in a time interval of length 1, is λ and the jump size is distributed according to F.

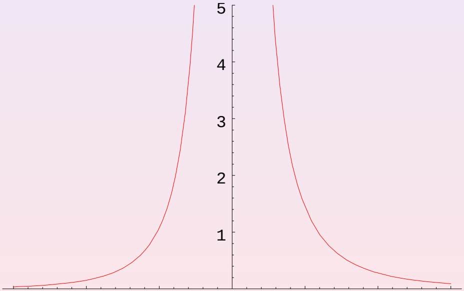
The Lévy measure

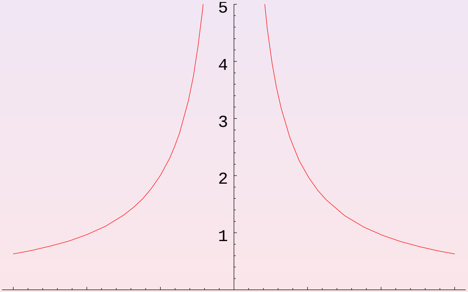
Distribution function of the Lévy measure of the Poisson process and the Density of the Lévy measure of a compound Poisson process with double-exponentially distributed jumps.



The Lévy measure

The density of the Lévy measure of an NIG and an α-stable process.





The Lévy measure

Proposition 6.1

Let L be a Lévy process with triplet $(b,\ c,\ \nu)$.

(1) If $\nu(\mathbb{R}) < \infty$ then almost all paths of L have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.

(2) If $\nu(\mathbb{R}) = \infty$ then almost all paths of L have an infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.

The Lévy measure

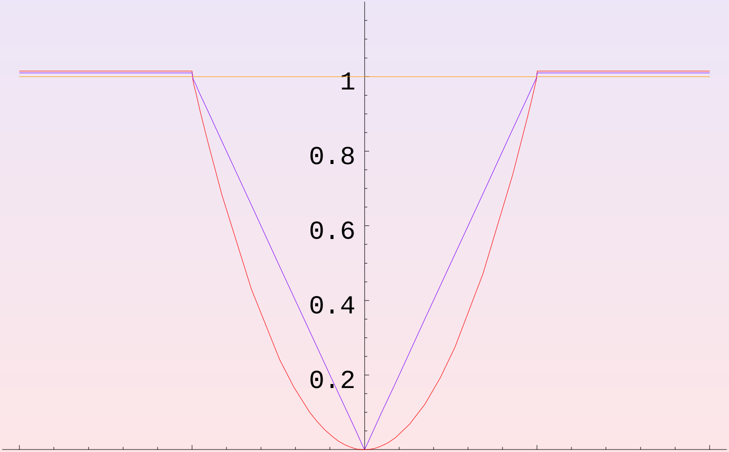
Proposition 6.2

Let L be a Lévy process with triplet $(b,\ c,\ \nu)$.

1. If $c = 0$ and $\displaystyle\int_{|x|\leq 1} |x|\nu(\mathrm{d}x) < \infty$ then almost all paths of L have finite variation.
2. If $c \neq 0$ or $\displaystyle\int_{|x|\leq 1} |x|\nu(\mathrm{d}x) = \infty$ then almost all paths of L have infinite variation.

The Lévy measure

The Lévy measure must integrate $|x|^2 \wedge 1$ (red); it has finite variation if it integrates $|x| \wedge 1$ (blue); it is finite if it integrates 1 (orange).



The Lévy measure

Proposition 6.3

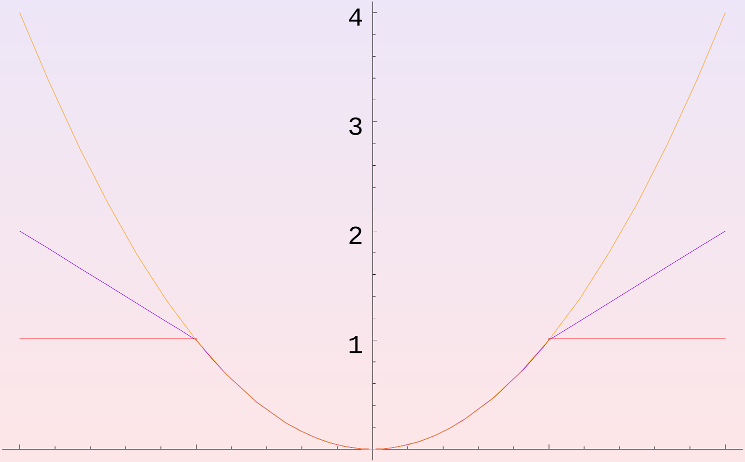
Let L be a Lévy process with triplet $(b,\ c,\ u)$. Then

(1) L_t has finite p-th moment for $p \in \mathbb{R}_+ (\mathrm{E}|L_t|^p < \infty)$ if and only if
$$\int_{|x|\geq 1} |x|^p \nu(\mathrm{d}x) < \infty.$$

(2) L_t has finite p-th exponential moment for $p \in \mathbb{R}(\mathrm{E}[e^{pL_t}] < \infty)$ if and only if
$$\int_{|x|\geq 1} e^{px} \nu(\mathrm{d}x) < \infty.$$

The Lévy measure

A Lévy process has first moment if the Lévy measure integrates $|x|$ for $|x| \geq 1$ (blue) and second moment if it integrates x^2 for $|x| \geq 1$ (orange).



Martingales and Lévy Processes

Proposition 10.1

Let $L = (L_t)_{t \geq 0}$ be a Lévy process with Lévy triplet $(b,\ c,\ \nu)$ and assume that $\mathrm{E}|L_t| < \infty$. L is a martingale if and only if $b = 0$.

Proposition 10.2

Let $L = (L_t)_{t \geq 0}$ be a Lévy process with Lévy exponent ψ and assume that $\mathrm{E}[e^{uL_t}] < \infty,\ u \in \mathbb{R}$. The process $M = (M_t)_{t \geq 0}$, defined as

$$M_t = \frac{e^{uL_t}}{e^{t\psi(u)}}$$

is a martingale.

Construction of Lévy Processes

(C1) Specifying a *Lévy triplet*

- Advantage: the characteristic function and the pathwise properties are known and allows the construction of a rich variety of models.
- Drawback: parameter estimation and simulation (in the infinite activity case) can be quite involved

(C2) Specifying an *infinitely divisible* random variable as the density of the increments at time scale 1 (i.e. L_1).

- Advantage: allows the easy estimation and simulation of the process.
- Drawback: the structure of the paths might be unknown

(C3) *Time-changing* Brownian motion with an independent increasing Lévy process.

- Advantage: allows for easy simulation
- Drawback: estimation might be quite difficult

Simulation

Finite Activity : Simulating the Lévy jump-diffusion

$$L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k$$

where $N_t \sim \mathrm{Poisson}(\lambda t)$ and $J \sim F(\mathrm{d}x)$, at fixed time points $t_1, \ldots, t_n$.

- simulate a standard normal variate
- transform it into a normal variate with variance $\sigma \triangle t$, where $\triangle t = t_i - t_{i-1}$ (denoted G_i)
- simulate a Poisson random variate with parameter $\lambda \triangle t$
- simulate the law of jump sizes J, i.e. simulate $F(\mathrm{d}x)$
- if the Poisson variate is larger than zero, add the value of the jump.

The discretized trajectory is

$$L_{t_{\mathrm{i}}} = bt_i + \sum_{j=1}^{i} G_i + \sum_{k=1}^{N_{t_i}} J_k .$$

Simulation

Infinite Activity : Simulating normal inverse Gaussian (NIG) process with parameters $\sigma,\ \theta,\ \kappa$ at fixed time points $t_1, \ldots, t_n$.

- simulate n independent inverse Gaussian variables I_i with parameters $\lambda_i = \frac{(\triangle t)^2}{\kappa}$ and $\mu_i = \triangle t$ where $\triangle t = t_i - t_{i-1}$ $i = 1\ldots, n$
- simulate n standard normal variables G_i
- set $\triangle L_i = \theta I_i + \sigma \sqrt{I_i} G_i$

The discretized trajectory is

$$L_{t_t} = \sum_{j=1}^{i} \triangle L_j.$$

Simulation

Infinite Activity : Simulating a variance gamma (VG) process with parameters $\sigma,\ \theta,\ \kappa$; at fixed time points $t_1, \ldots, t_n$.

- simulate n independent gamma variables Γ_i with parameter $\dfrac{\triangle t}{\kappa}$ where
$$\triangle t = t_i - t_{i-1},\ i = 1,\ \ldots\ n$$
- set $\Gamma_i = \kappa \Gamma_i$
- simulate n standard normal variables G_i
- set $\triangle L_i = \theta \Gamma_i + \sigma \sqrt{\Gamma_i} G_i$

The discretized trajectory is

$$L_{t_{\mathrm{t}}} = \sum_{j=1}^{i} \triangle L_j.$$

Popular Models

Generalized Hyperbolic (Eberlein and Prause 2002)

$$L_1 \sim \mathrm{GH}(\alpha,\ \beta,\ \delta,\ \mu,\ \lambda)$$

Density:

$$f_{GH}(x) = c(\lambda,\ \alpha,\ \beta,\ \delta)(\delta^2 + (x-\mu)^2)^{(\lambda-\frac{1}{2})/2}$$

$$\times K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2 + (x-\mu)^2})\exp(\beta(x-\mu)),$$

where

$$c(\lambda,\ \alpha,\ \beta,\ \delta) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}$$

and K_λ denotes the Bessel function of the third kind with index λ

Popular Models

Generalized Hyperbolic (Eberlein and Prause 2002)
Parameters:

- $\alpha > 0$ determines the shape
- $0 \leq |\beta| < \alpha$ determines the skewness
- $\mu \in \mathbb{R}$ the location
- $\delta > 0$ is a scaling parameter
- $\lambda \in \mathbb{R}$ affects the heaviness of the tails
    - $\lambda - 1$ we get the hyperbolic distribution
    - $\lambda = -\frac{1}{2}$ we get the normal inverse Gaussian (NIG)

Popular Models

Generalized Hyperbolic (Eberlein and Prause 2002)
Characteristic Function:

$$\varphi_{GH}(u) = e^{iu\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\frac{\lambda}{2}}\frac{K_\lambda(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})},$$

First moment:

$$\mathrm{E}[L_1] = \mu + \frac{\beta\delta^2}{\zeta}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}$$

Second moment:

$$\mathrm{Var}[L_1] = \frac{\delta^2}{\zeta}\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + \frac{\beta^2\delta^4}{\zeta^2}\left(\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \frac{K^2_{\lambda+1}(\zeta)}{K^2_\lambda(\zeta)}\right),$$

with $\zeta = \delta\sqrt{\alpha^2-\beta^2}$
Lévy triplet: $(E[GH],\ 0,\ \nu^{GH})$

Popular Models

Normal Inverse Gaussian (Barndorff-Nielsen 1997)
(GH with $\lambda = -1/2$)
Density:

$$f_{NIG}(x) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2-\beta^2} + \beta(x-\mu)\right) \frac{K_1(\alpha\delta\sqrt{1+(\frac{x-\mu}{\delta})^2})}{\sqrt{1+(\frac{x-\mu}{\delta})^2}},$$

Characteristic Function:

$$\varphi_{NIG}\ (u) = e^{iu\mu} \frac{\exp(\delta\sqrt{\alpha^2-\beta^2})}{\exp(\delta\sqrt{\alpha^2-(\beta+iu)^2})}.$$

First Moment:

$$\mathrm{E}[L_1] = \mu + \frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}$$

Second Moment:

$$\mathrm{Var}[L_1] = \frac{\delta}{\sqrt{\alpha^2-\beta^2}} + \frac{\beta^2\delta}{\left(\sqrt{\alpha^2-\beta^2}\right)^3}$$

Asset Price Model

Real-world measure

We model the asset price process as the exponential of a Lévy process

$$S_t = S_0 \exp L_t, \quad 0 \leq t \leq T$$

where, L is the Lévy process whose infinitely divisible distribution has been estimated from the data set available for the asset.

Risk-neutral measure

We model the asset price process as the exponential of a Lévy process

$$S_t = S_0 \exp L_t, \quad 0 \leq t \leq T$$

where, the Lévy process L has the triplet $(\bar{b}, \bar{c}, \bar{\nu})$ and has a finite first moment and exponential moment. L then has the canonical decomposition

$$L_t = \bar{b}t + \sqrt{\bar{c}}\bar{W}_t + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^L)(\mathrm{d}s, \mathrm{d}x)$$

with

$$\bar{b} = r - \delta - \frac{\bar{c}}{2} - \int_{\mathbb{R}} (\mathrm{e}^x - 1 - x)\bar{\nu}\mathrm{d}x$$

Pricing Options

Three predominant methods:

- Transform Methods
  - Simple and fast
  - Exotic options cannot be handled easily
- PIDE Methods
  - Complex and Exotic options can be treated easily
  - Slower speed compared to transform methods and increased computational complexity when handling options on several assets
- Monte Carlo Methods
  - Options on several assets can be treated easily
  - Slow computational speed

Transform Methods (Carr and Madan 1999)

Let $\mathbb{Q}$ be a risk-neutral measure and $q_T(s_T)$ be the risk-neutral density for the log price.
The call value is then the discounted expected value of the payoff under $\mathbb{Q}$,

$$C_T(k) = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(\mathrm{e}^{s_T} - \mathrm{e}^{k}\right)_+\right] = \int_k^\infty \mathrm{e}^{-rT}(\mathrm{e}^{u} - \mathrm{e}^{k})q_T(u)\mathrm{d}u$$

Note: If $\mathrm{e}^k \to 0$ we have $k \to -\infty$ and hence $C_T \to S_0$. So $C_T(k)$ is not square integrable!

Transform Methods (Carr and Madan 1999)

To make $C_T(k)$ square-integrable Carr and Madan introduce a damping factor

$$c_T \equiv \exp(\alpha k) C_T(k)$$

Idea: $\alpha > 0$ causes $c_T(k)$ to decay as $k \to -\infty$
Carr and Madan found that if $\mathbb{E}[S_T^{\alpha+1}]$ then $c_T(k)$ is square integrable
Fourrier transform:

$$\psi_T(v) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}vk} c_T(k) \mathrm{d}k$$

Reversing the transform and undamping:

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}vk} \psi_T(v) \mathrm{d}v$$

$$\psi_T(v) = \frac{\mathrm{e}^{-rT} \phi_T(v - (\alpha+1)\mathrm{i})}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v}$$

Pricing European Options

Valuation of European options using Laplace transforms (Raible)
Assumptions:

- (D1) Assume that $\varphi_{L_T}(z)$, the extended characteristic function of L_T, exists for all $z \in \mathbb{C}$ with $\Im z \in I_1 \supset [0,1]$.
- (D2) Assume that P_{L_T}, the distribution of L_T, is absolutely continuous w.r.t. the Lebesgue measure λ with density ρ.
- (P1) Consider a European-style payoff function $f(S_T)$ that is integrable.
- (P2) Assume that $x \mapsto \mathrm{e}^{-Rx}|f(\mathrm{e}^{-x})|$ is bounded and integrable for all $R \in I_2 \subset \mathbb{R}$.
- (B1) Assume that $I_1 \cap I_2 \neq \emptyset$.

Pricing European Options (Raible)

Theorem B.1.

Let F_1 and F_2 be measurable complex-valued functions on the real line. If $|F_1(x)|$ is bounded and if $F_2(x)$ is absolutely integrable, then the convolution $F_1 * F_2$, defined by

$$F_1 * F_2(x) := \int_{\mathbb{R}} F_1(x-y) F_2(y) dy,$$

is a well-defined function on $\mathbb{R}$. $F_1 * F_2$ is bounded and uniformly continuous.

Pricing European Options (Raible)

Theorem B.2.

Let F_1 and F_2 be measurable complex-valued functions on the real line. Let $z \in \mathbb{C}$ and $R := \Re(z)$. If

$$\int_{\mathbb{R}} e^{-Rx}|F_1(x)|dx < \infty \text{ and } \int_{\mathbb{R}} e^{-Rx}|F_2(x)|dx < \infty,$$

and if $x \mapsto e^{-Rx}|F_1(x)|$ is bounded, then the convolution $F(x) := F_1 * F_2(x)$ exists and is continuous for all $x \in \mathrm{B}$, and we have

$$\int_{\mathbb{R}} e^{-Rx}|F(x)|dx < \infty \text{ and}$$

$$\int_{\mathbb{R}} e^{-zx}F(x)dx = \int_{\mathbb{R}} e^{-zx}F_1(x)dx \cdot \int_{\mathbb{R}} e^{-zx}F_2(x)dx.$$

Pricing European Options (Raible)

Theorem B.3.

Let F be a measurable complex-valued function on the real line. Let $R \in \mathbb{R}$ such that

$$f(z) = \int_{\mathbb{R}} e^{-zx} F(x) dx \; (z \in \mathbb{C} \; \Re(z) = R),$$

with the integral converging absolutely for $z = R^1$ Let $x \in \mathrm{B}$ such that the integral

$$\int_{R-i\infty}^{R+i\infty} e^{zx} f(z) dz$$

exists as a Cauchy principal value. Assume that F is continuous at the point x. Then

$$F(x) = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} e^{zx} f(z) dz,$$

where the integral is to be understood as the Cauchy principal value if the integrand is not absolutely integrable.

Pricing European Options

Definition 15.1

Let $L_h(z)$ denote the bilateral Laplace transform of a function h at $z \in \mathbb{C}$, i.e. let

$$L_h(z) := \int_{\mathbb{R}} \mathrm{e}^{-zx} h(x) dx.$$

Transform Methods (Raible)

The value of a European style option with payoff $f(S_T)$

$$V = \mathrm{e}^{-rT} \int_{-\infty}^{\infty} f(S_0 \mathrm{e}^{x}) q(x) \mathrm{d}x$$

Consider the modified payoff function $g(x) := f(\mathrm{e}^{-x})$ and let $\zeta = -\ln(S_0)$

$$f(S_0 \mathrm{e}^{x}) = f(\mathrm{e}^{-\zeta} \mathrm{e}^{x}) = f(\mathrm{e}^{-(\zeta - x)}) = g(\zeta - x)$$

Rewriting we have

$$V = \mathrm{e}^{-rT} \int_{-\infty}^{\infty} g(\zeta - x) q(x) \mathrm{d}x = \mathrm{e}^{-rT} (g * q)(\zeta).$$

Transform Methods (Raible)

Take an $R \in \mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \mathrm{e}^{-Rx} |g(x)| < \infty$$

where $\mathrm{e}^{-Rx}|q(x)|$ is bounded and

$$\int_{-\infty}^{\infty} \mathrm{e}^{-Rx} |q(x)| < \infty.$$

With $u \in \mathbb{R}$ and the value of the option written as a convolution we can then take the Laplace transform to get the product of Laplace transforms

$$\mathcal{L}_V(R + \mathrm{i}u) = \mathrm{e}^{-rT} \mathcal{L}_g(R + \mathrm{i}u) \mathcal{L}_q(R + \mathrm{i}u)$$

Finally we can take the inverse Laplace transform to get

$$V(\zeta) = \frac{\mathrm{e}^{\zeta R - rT}}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}u\zeta} \mathcal{L}_g(R + \mathrm{i}u) \phi_{L_T}(\mathrm{i}R - u) \mathrm{d}u.$$

Transform Methods (Raible)

Theorem 15.2

Assume that the above mentioned conditions are in force and let $g(x) := f(\mathrm{e}^{-x})$ denote the modified payoff function of an option with payoff $f(x)$ at time T. Choose an $R \in I_1 \cap I_2$. Letting $V(\zeta)$ denote the price of this option, as a function of $\zeta := -\log S_0$, we have

$$(15.1) \qquad V(\zeta) = \frac{\mathrm{e}^{\zeta R - rT}}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{iu\zeta} L_g(R + iu) \varphi_{L_T}(iR - u) du,$$

whenever the integral on the r.h.s. it exists (at least as a Cauchy principal value).

Transform Methods (Raible)

European Call:

$$(15.3) \qquad L_g(z) = \frac{1}{z(z+1)}$$

for $z \in \mathbb{C}$ with $\Re z = R \in I_2 = (-\infty,\ -1)$.
European Put: $z \in \mathbb{C}$ with $\Re z = R \in I_2 = (0,\ \infty)$.
European Digital Call $(f(S_T) = \mathbb{1}_{\{S_T > K\}})$:

$$(15.4) \qquad L_g(z) = -\frac{1}{z}\left(\frac{K}{S_0}\right)^z$$

for $z \in \mathbb{C}$ with $\Re z = R \in I_2 = (-\infty,\ 0)$.
European Digital Put $(f(S_T) = \mathbb{1}_{\{S_T < K\}})$:

$$(15.5) \qquad L_g(z) = \frac{1}{z}\left(\frac{K}{S_0}\right)^z$$

for $z \in \mathbb{C}$ with $\Re z = R \in I_2 = (0,\ \infty)$.

PIDE methods

Let us denote by $G(S_t, t)$ the time-t price of a European option with payoff function g on the asset S_t the price is given by

$$G(S_t, t) = \mathrm{e}^{-r(T-t)}\mathbb{E}[g(S_T)] =: V_t, \ 0 \leq t \leq T$$

By arbitrage theory, we know that the discounted option price process must be a martingale under a martingale measure. Therefore, any decomposition of the price process as

$$\mathrm{e}^{-rt} V_t = V_0 + M_t + A_t$$

where $M \in \mathcal{M}_{\mathrm{loc}}$ and $A \in \mathcal{A}_{\mathrm{loc}}$, must satisfy $A_t = 0$ for all $t \in [0, \ T]$. This condition yields the desired PIDE.

PIDE methods

NOW, for notational but also computational convenience, we work with the driving process L and not the asset price process S, hence we derive a PIDE involving $f(L_t,\ t) := G(S_t,\ t)$, or in other words

$$f(L_t,\ t) = \mathrm{e}^{-r(T-t)}\mathrm{E}[g(S_0\mathrm{e}^{L_T})] = V_t,\ 0 \leq t \leq T.$$

PIDE methods

Assume that $f \in C^{2,1}(\mathbb{R} \times [0, T])$, An application of Itô's formula yields:

$$\begin{aligned}
& \mathrm{d}(\mathrm{e}^{-rt} V_t) \\
= \; & \mathrm{d}(\mathrm{e}^{-rt} f(L_{t-}, \; t)) \\
= \; & \mathrm{e}^{-rt}\{-rf(L_{t-}, \; t)\mathrm{d}t + \partial_2 f(L_{t-}, \; t)\mathrm{d}t + \partial_1 f(L_{t-}, \; t)b\mathrm{d}t \\
& + \partial_1 f(L_{t-}, \; t)\sqrt{c}\mathrm{d}W + \int_{\mathbb{R}} \partial_1 f(L_{t-}, \; t)z(\mu^L - \nu^L)(\mathrm{d}z, \; \mathrm{d}t) \\
& + \frac{1}{2}\partial_1^2 f(L_{t-}, \; t)c\mathrm{d}t \\
& + \int_{\mathbb{R}} (f(L_{t_-} + z, \; t) - f(L_{t-}, \; t) - \partial_1 f(L_{t_-}, \; t)z)(\mu^L - \nu^L)(\mathrm{d}z, \; \mathrm{d}t) \\
& + \int_{\mathbb{R}} (f(L_{t_-} + z, \; t) - f(L_{t-}, \; t) - \partial_1 f(L_{t_-}, \; t)z)\nu(\mathrm{d}z)\mathrm{d}t\}
\end{aligned}$$

PIDE methods

The bounded variation part vanishes identically. Hence, the price of the option satisfies the partial integro-differential equation

$$0 = -rf(x,\ t) + \partial_2 f(x,\ t) + \partial_1 f(x,\ t)b + \frac{c}{2}\partial_1^2 f(x,\ t)$$

$$+ \int_{\mathbb{R}} (f(x+z,\ t) - f(x,\ t) - \partial_1 f(x,\ t)z)\nu(\mathrm{d}z),$$

for all $(x,\ t) \in \mathbb{R} \times (0,\ T)$, subject to the terminal condition

$$f(x,\ T) = g(\mathrm{e}^x).$$

Monte Carlo methods

The payoff of the call option with strike K at the time of maturity T is $g(S_T) = (S_T - K)^+$ and the price is provided by the discounted expected payoff under a risk-neutral measure, i.e.

$$C_T(S,\ K) = \mathrm{e}^{-rT}\mathrm{E}[(S_T - K)^+].$$

Simulate the terminal value of asset price $S_T = S_0 \exp L_T$. Let S_{T_k} for $k = 1, \ldots, N$ denote the simulated values; then, the option price $C_T(S,\ K)$ is estimated by the average of the prices for the simulated asset values, that is

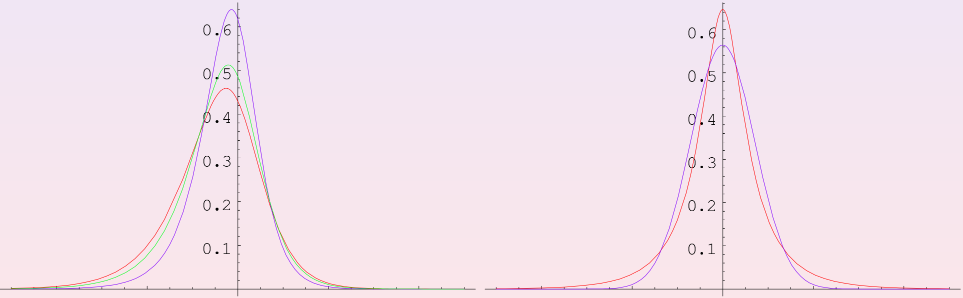
$$\hat{C}_T(S,\ K) = \mathrm{e}^{-rT}\sum_{k=1}^{N}(S_{T_k} - K)^+,$$

and by the Law of Large Numbers we have that

$$\hat{C}_T(S,\ K) \to C_T(S,\ K) \text{ as } N \to \infty.$$

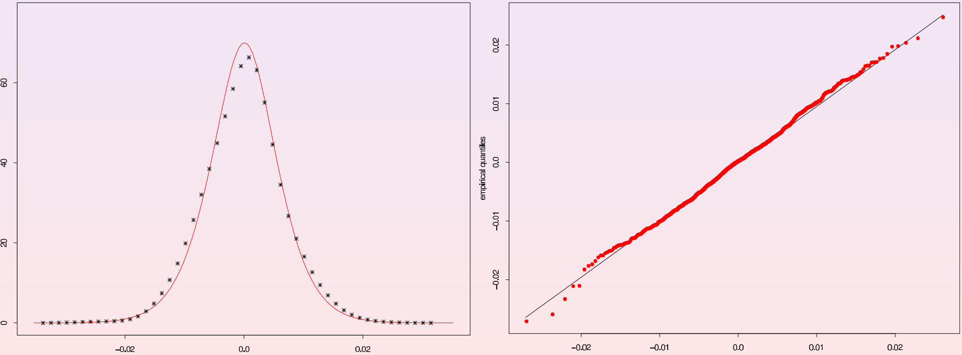
Empirical Evidence

Densities of hyperbolic (red), NIG (blue) and hyperboloid distribution (left). Comparison of the GH (red) and Normal distributions (with equal mean and variance).



Empirical Evidence

Empirical distribution and Q-Q plot of EUR/USD daily log-returns with fitted GH (red).



Empirical Evidence

Implied volatilities of EUR/USD options and calibrated NIG smile.