HEDGING STRATEGIES FOR CANADA'S OIL
Application of CTOs driven by a Copula

Paul Obour

University of Calgary
Department of Mathematics & Statistics
Lunch at the Lab Seminar

November 16, 2011

UNIVERSITY OF CALGARY

Outline

1. Introduction - Currency Translated Options
   - What are CTOs
   - Applications of CTOs

Outline

1. Introduction - Currency Translated Options
   - What are CTOs
   - Applications of CTOs
2. Pricing of CTOs
   - Underlying Components
   - Steps in Pricing CTOs
   - Pricing the CTO Types

Outline

UNIVERSITY OF CALGARY

Outline

Outline

UNIVERSITY OF CALGARY

Definition and Features

- Options on foreign assets where the payoff is exchanged into domestic currency at expiration.
- CTOs are offered with a variety of different features:
  1. A foreign equity call struck in foreign currency
  2. A foreign equity call struck in domestic currency
  3. An equity linked foreign exchange call
  4. A fixed exchange rate foreign equity call

Types of CTOs

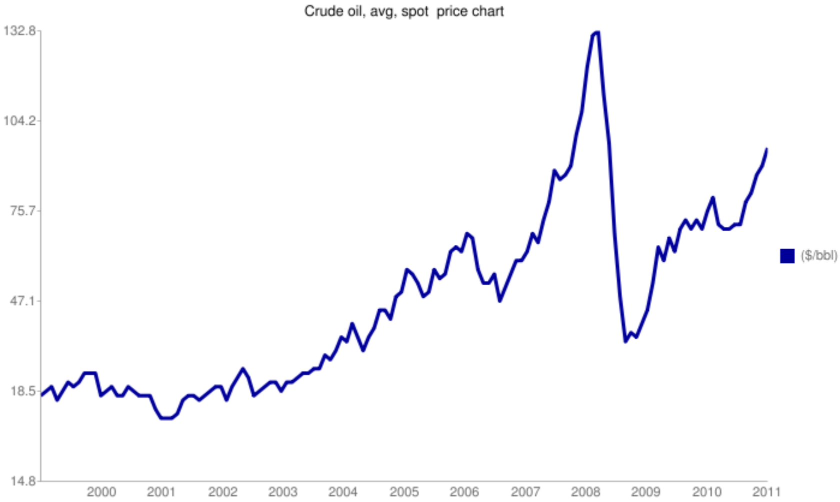
- The contingent claim is affected by the co-movement of the underlying asset price and the movement in some **FX**.
- CTOs are of three types:
  1. The simplest CTO is the flexible exchange rate version - Flexos
  2. The second and third version of the product are Compos - Composite or Joint Options.
  3. The true Quanto - Quantity Adjusted Options

Applying CTOs

- In theory, can exist for any asset or liability denominated in a foreign currency.
- CTOs have been developed to eliminate risks associated with foreign investments:
  - risk in changing prices
  - exchange rate risk
- An example is the Chicago Merchantile Exchange's Nikkei 225 stock index contract, which uses the nominal price of the Yen-denominated index applied to as US Dollar notional principal (**25/9/1990**).

UNIVERSITY OF CALGARY

Crude Oil Price Path 2000 - 2011

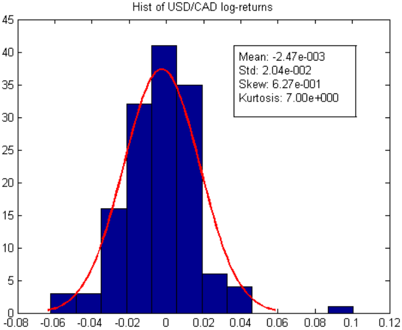


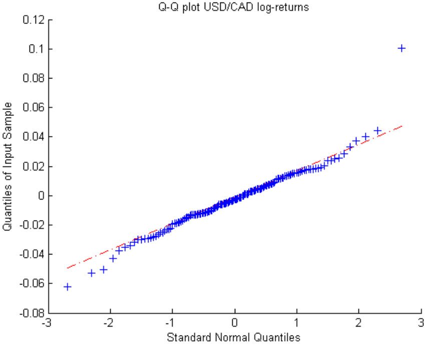
Path of USD/CAD Futures 2007 - 2009

Black & Scholes Assumptions

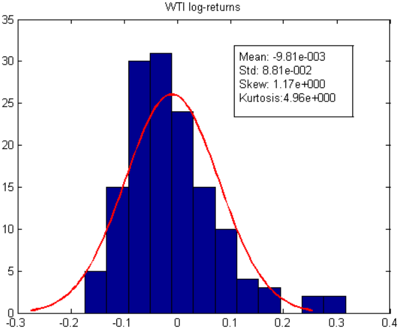
1. The stock pays no dividends during the option's life.
2. European exercise terms are used.
3. Markets are efficient.
4. No commissions are charged.
5. Interest rates remain constant over the lifetime of the option and are quoted in a continuously compounded annual rate.
6. Returns are lognormally distributed.

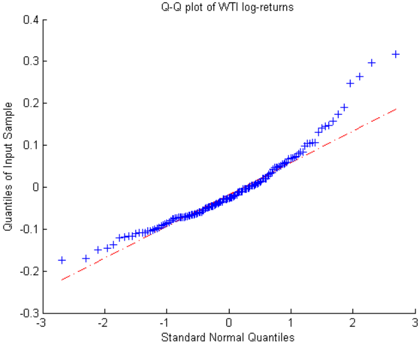
USD/CAD Log-returns 2000 - 2011





WTI Log-returns 2000 - 2011





Descriptive Statistics & Distribution Test

- The normal distribution is symmetric (zero skew) with an expected kurtosis of 3.
- The WTI appears to have a skewed distribution whiles the FX log-returns would be appropriate for the normal distribution.
- If log-returns are normal, the sample quantiles will lie close to the line $y = x$.
- Correlation for the futures prices (WTI & FX) can get as high as 0.9368 and 0.4290 for their respective log-returns.

UNIVERSITY OF CALGARY

Exotic Option Pricing

- Evaluate the option without the currency impact:
  1. Black Scholes (1973) - European options on non-dividend paying stock
  2. Merton (1973) - extended to continuous dividend paying shares
- Adjustments must be made due to the currency effects once the standard option valuation is completed.
  1. The correlation between the underlying stock and **FX**
  2. The difference in the interest rates in the two currencies.

UNIVERSITY OF CALGARY

Pricing the Flexos

- The terminal payoff:

$$V_d^{(1)}(S,T) = F_0.\max(S_T - X_f, 0)$$

- The corresponding price formula is:

$$V_d^{(1)}(S,t) = F_0.e^{-r_f\tau}\left[Se^{\delta_S^d\tau}N(d_1) - X_fN(d_2)\right], \quad \tau = T - t$$

where

$$d_1 = \frac{\ln\frac{S}{X_f} + (\delta_S^d + \frac{\sigma_S^2}{2})\tau}{\sigma_S\sqrt{\tau}}, \quad d_2 = d_1 - \delta_S\sqrt{\tau}$$

- The price formula does not depend on the exchange rate F.

UNIVERSITY OF CALGARY

Pricing the Compos

- The **Compos** comes in two varieties:
  1. Call on foreign equity denominated in domestic currency
  2. Floating exchange rate foreign equity call
- The terminal payoff is:

$$V_d^{(2)}(S_T, F_T, T) = \max(F_T S_T - X_d, 0) = \max(S_T^* - X_d, 0)$$

- This is an option to exchange one risky asset for another. Margrabe (1978) solved this as a generalization of the BS.

$$\mathbf{V(S(0), T) = e^{-q_1 T} S_1(0) N(d_1) - e^{-q_2 T} S_2(0) N(d_2)}$$

- Hence the corresponding price formula is seen to be:

$$V_d^2(S^*, t) = S e^{-q\tau} N(\hat{d}_1) - X_d e^{-r_d \tau} N(\hat{d}_2), \quad \tau = T - t,$$

UNIVERSITY OF CALGARY

Pricing the Quanto Option

- The Quanto fixes at inception the **FX** at which the proceeds of a foreign equity call will be paid.
- Pricing incorporates both the correlations between the stock price & **FX** and the interest rate differentials of the two countries.
- As these effects introduce complications, neither the standard Black Scholes nor the generalisation of Black Scholes by Margrabe is applicable.

  Holder:

$$\begin{aligned} V_d^3(S_T, F_T, T) &= F_0.\max(0, S_T - X_f) \\ &= \max(0, [S_T.F_T] - X_f.F_T) \end{aligned}$$

  Seller:

$$V_d^{3\prime}(S_T, F_T, T) = F_0/F_T.\max(0, S_T - X_f)$$

CTOs Hedging Structure

- **Flexos** are hedged in exactly the same manner as standard European Options.
  Compos structures:
- Margrabe (1978) proposed a similar hedging strategy to the **Black & Scholes** approach since we have the option to exchange one risky asset for the other.
- The difference in the Margrabe case is that:
  - there are two Deltas that must be estimated, and
  - each of the risky assets must be dynamically hedged.
- Hedge performance of these options is extremely sensitive to stochastic correlations.

UNIVERSITY OF CALGARY

Hedging Quantos

- Quantos are second order correlation dependent securities: the correlation risks are between FX and the option premium not the underlying equity holding.
- There is no appreciable difference between the standard call and the "Quantoed" call when the correlation is **zero** and the interest rate differential is also zero.
- The zero interest rate differential means that the expected forward price and the spot price are equal.
- Hence, we will hedge the risk of the USD value of the premium with the forward contract of CAD for USD on the expiration date of the "Quantoed" product.

Hedging Quantos

- A Quanto can be **replicated** by a portfolio containing stock (CAD), USD cash and CAD borrowing as follows:
-
$$f_{Quanto} = \Delta_S S + \Delta_{\frac{1}{e_0}} \frac{1}{e_0} - V(S, \frac{1}{e_0})$$

where $\Delta_{\frac{1}{e_0}} = \frac{\partial f_{Quanto}}{\partial \frac{1}{e_0}} = e_0 . f_{Quanto}$

- This is the amount of USD cash held in the hedge (the current value of the option in USD).
- So the premium is retained in USD dollars and we get:

$$\Delta_S S = V(S, \frac{1}{e_0})$$

UNIVERSITY OF CALGARY

CTOs Hedging Difficulties

- With Compos, hedge performance suffered when the correlation were not constant.
- Most practitioners choose to assume that the correlation between the asset and the FX is zero.
- However, with $\rho = 0$, market practitioners often value American Call Options using European models.
  - This tends to work if the dividend yields are fairly low and the option is "out-of or at-the-money".
- But if the option is "Quantoed" into a high interest currency, European and American prices can diverge sharply.
- Existence of "sticky" or implied volatility surface.

What are Lévy Processes?

- Given a probability space $(\Omega, \mathbf{F}, P)$, a stochastic process L_t is a Lévy process if it has independent and stationary increments and has a stochastically continuous path, i.e. for any $\epsilon > 0$, $\lim_{s \to t} P(|L_t - L_s| > \epsilon) = 0$.
- The simplest possible LP are the standard Brownian motion W_t, Poisson processes N_t, and compound Poisson processes $\sum_{i=1}^{N_t} Y_i$, where Y_i are i.i.d. random variables.
- There are two types of LPs, jump-diffusion and infinite activity Lévy process.

Characteristic Function & Sample Paths

- The **Lévy-Khintchine formula** describes the distribution of each independent increment and can be written in terms of (γ, σ^2, ν) as

$$\frac{1}{t}\log \mathbb{E}\left[e^{iuL_t}\right] = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{R^d}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right)\nu(dx)$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}\setminus\{0\}$

- The structure of the sample paths of L_t is represented using the **Lévy-Itô Decomposition**.

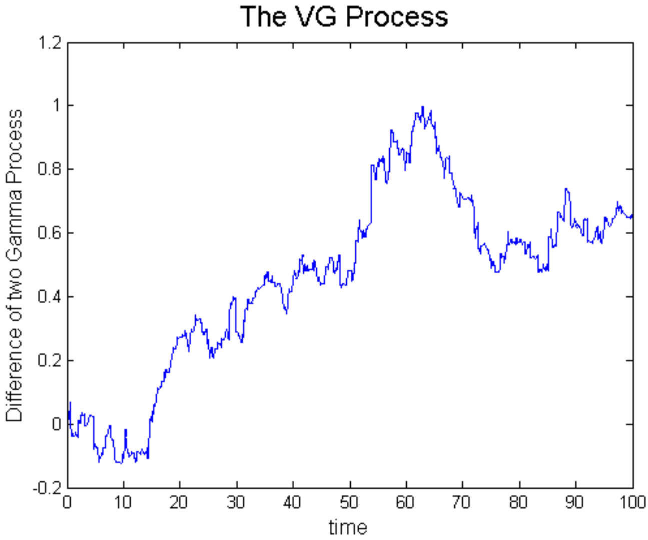
$$L_t = \gamma t + B_{\sigma^2}(t) + \int_{|x|<1} x\tilde{N}_t(dx) + \int_{|x|\geq 1} xN_t(dx)$$

UNIVERSITY OF CALGARY

Desirable Properties of Lévy Processes

- LPs are more versatile than Gaussian driven processes as they can model skewness, excess kurtosis and jumps in the data.
- If we are interested in relaxing the normality assumption of the log-returns, then we use LP.
- LP provide us with the appropriate framework to adequately describe all these observations, both in the 'real' and the "risk-neutral" world.

Sample path of Lévy processes



Pricing Flexos under Lévy Processes

- Consider a market with an FX and a foreign asset (F,S) given by $F_t = F_0 e^{L_t^1}$, $S_t = S_0 e^{L_t^2}$, where (L_t^1, L_t^2) is a two dimensional LP.
- We define the contingent claim as $\Phi_T = F_T(S_T - X_f)^+$
- Thus a European Call is evaluated by

$$V_{X_f,T}(F,S) = e^{-r_d T}\mathbb{E}\left[F_T(S_T - X_f)^+ | \mathcal{F}_0\right]$$

- Incorporate the FX into a pricing measure. Let $V = \{\omega \in \Omega : S_T > X_f\}$, and let $\beta = -\log(\mathbb{E}^Q\left[e^{L_t^1}\right])$.
- Define a new martingale measure $\tilde{Q}$ equivalent to Q by $\frac{d\tilde{Q}_T}{dQ_T} = \exp\{L_{t^1} + \beta T\}$, then $d\tilde{Q}_T = \frac{e^{L_t^1}}{\mathbb{E}^Q\left[e^{L_t^1}\right]} dQ_T$.

UNIVERSITY OF CALGARY

Pricing Flexos under Lévy Processes

- Using this new measure, the evaluation formula becomes

$$
\begin{aligned}
C_{K_f,T}(F,S) &= e^{-r_d T}\int_C F_T(S_T - X_f)\,dQ \\
&= e^{-r_d T}\int_C F_0 \mathbb{E}^Q\left[e^{L^1_T}\right](S_T - X_f)\,d\tilde{Q} \\
&= e^{-r_d T}e^{-\beta T}F_0\int_C (S_T - X_f)\,d\tilde{Q} \\
&= e^{-(r_d+\beta)T}F_0\mathbb{E}^{\tilde{Q}}\left[(S_T - X_f);C\right] \\
&= e^{-(r_d+\beta)T}F_0\mathbb{E}^{\tilde{Q}}\left[(S_T - X_f)^+\right].
\end{aligned}
$$

- Unlike the Gaussian case, the discounting rate in the non-Gaussian model above is $(r_d + \beta)$

UNIVERSITY OF CALGARY

Copulas: Definition

- A 2-*dimensional copula* is a real function C: $[0,1]^2 \to [0,1]$ with the following properties:
  1. $\forall\, u \in [0,1]\,, C(0,u) = C(u,0) = 0,$
  2. $\forall\, u \in [0,1]\,, C(u,1) = u$ and $C(1,u) = u.$
  3. $\forall\, (u_1,u_2),(v_1,v_2) \in [0,1] \times [0,1]$ with $u_1 \leq v_1$ and $u_2 \leq v_2$:

$$C(u_2,v_2) - C(u_1,u_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0$$

- Thus a copula is a multivariate distribution with support in $[0,1]^n$ and with uniform marginals.
- **Sklar's theorem** Let $F_{x,y}$ be a joint distribution function with continuous marginals $F_1(x)$ and $F_2(y)$. Then there exists a unique copula C such that

$$F(x,y) = C(F_1(x), F_2(y))$$

Some Families of Copulas

- **Elliptical copulas** are derived from multivariate elliptical distributions.
  1. The traditonal use of correlation to model dependence implies using the **Gaussian copula** which has cdf:

  $$C_n^{\Phi}(\mathbf{u};\Omega^{\Phi}) = \Phi_n(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n);\Omega^{\Phi})$$

  2. The **Student's t copula** is closely related to the Gaussian copula, with cdf:

  $$C_n^{\Psi}(\mathbf{u};\Omega^{\Psi},\nu) = \Psi_n(\Psi^{-1}(u_1,\nu),\ldots,\Psi_{-1}(u_n;\nu);\Omega^{\Psi},\nu)$$

- The Student's t copula exhibits tail dependence (even if correlation coefficients equal zero).

Some Families of Copulas

- The function $C(U,V) = \psi^{[-1]}(\psi(u) + \psi(v))$ is an **Archimedean copula** with generator ψ if and only if $\psi^{[-1]}$ is n-monotonic:

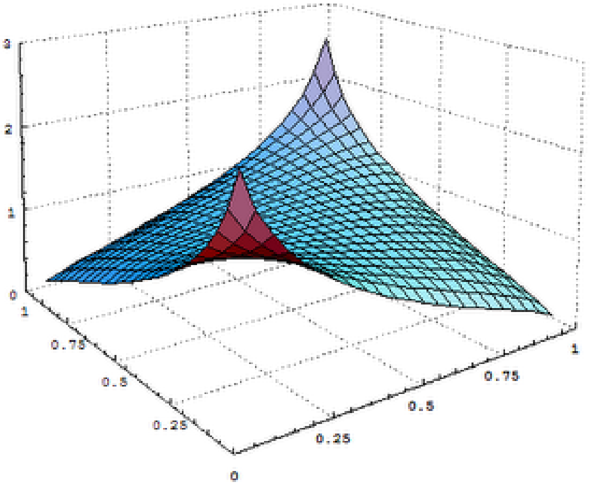
$$(-1)^k \frac{d^k \psi^{[-1]}(t)}{dt^k} \geq 0, \quad \forall k = 0, 1, \ldots, n.$$

- $\psi^{[-1]}$ is said to be completely monotonic if the above relation holds for all $n \in N$.
- The following are examples of copulas in the Archimedean family:
  - **Clayton's copula**, which plays the role of a limit copula.
  - **Gumbel's copula**, plays a special role in the description of dependence using extreme value theory.

Why Copulas?

- Non-linear dependence
- Be able to measure dependence for heavy tail distributions
- Very flexible: parametric, semi-parametic or non-parametric
- Be able to study asymptotic properties of dependence structures
- Can be more probabilistic or more statistical.

Nature of a Gaussian Copula (error function)



UNIVERSITY OF CALGARY

Hedging with Copulas - The Bivariate Case

- We first define the Frechet bounds:
$\max(F_1(x)+F_2(y)-1,0) \le F(x,y) \le \min(F_1(x),F_2(y))$
- We replicate and price two single digital options with the same exercise date written on the underlying markets S_1 and S_2 for strikes K_1 and K_2 respectively.

A breakdown of the Sample space

S$_1 \backslash S_2$	State H	State L
State H	$S_1 \ge K_1, S_2 \ge K_2$	$S_1 \ge K_1, S_2 < K_2$
State L	$S_1 < K_1, S_2 \ge K_2$	$S_1 < K_1, S_2 < K_2$

- The bivariate digital option pays 1 unit if both assets are in state H.

UNIVERSITY OF CALGARY

Hedging with Copulas - The Bivariate Case

- We denote with P_1, P_2 and B the prices of the single digital options and the risk-free asset resp.
- *Proposition*: The no-arbitrage price $P(S_1 \geq K_1, S_2 \geq K_2)$ of a bivariate digital option is bounded by the inequality

$$\max(P_1 + P_2 - B, 0) \leq P(S_1 \geq K_1, S_2 \geq K_2) \leq \min(P_1, P_2)$$

- Then we have:

$$\max\left(\frac{P_1}{B} + \frac{P_2}{B} - 1, 0\right) \leq \frac{P(S_1 \geq K_1, S_2 \geq K_2)}{B} \leq \min\left(\frac{P_1}{B}, \frac{P_2}{B}\right)$$

- These bounds emerged from no-arbitrage considerations only.

UNIVERSITY OF CALGARY

Hedging with Copulas - The Bivariate Case

- We are lead to a function of the kind:

$$\frac{P(S_1 \geq K_1, S_2 \geq K_2)}{B} = C\left(\frac{P_1}{B}, \frac{P_2}{B}\right)$$

- *Proposition*: The bivariate pricing kernel is a function $C(u, v)$ taking the univariate pricing kernels as arguments. The function must fulfill the requirements of a copula in order to rule out arbitrage opportunities.
- The arbitrage-free pricing kernel has a super-replication strategy as:

$$\max\left(\frac{P_1}{B} + \frac{P_2}{B} - 1, 0\right) \leq C\left(\frac{P_1}{B}, \frac{P_2}{B}\right) \leq \min\left(\frac{P_1}{B}, \frac{P_2}{B}\right)$$

What is next?

1. To consider pricing and hedging Quantos using Lévy processes.
2. Use copulas to capture the correlations in data more efficiently and to implement the hedge for bivariate options in an arbitrage-free setting.
3. Overcome the difficulties in tail behaviours by using the best copula.
4. Work around different sets of data to try and eliminate the risk due to model selection.

References

- David Applebaum (2009)
  Lévy Processes and Stochastic Calculus
  Cambridge University Press, 2nd edition, pp: 43-78
- Roger B. Nelsen (1999)
  Lecture Notes in Statistics - An Introduction to Copulas
  Springer Science + Business Medi Inc., v. 139
- Cherubini, U., Luciano, E and Vecchiato, W(2004)
  Copula Methods in Finance
  John Wiley & Sons, Ltd
- Cont, R and Tankov, P (2004)
  Financial Modelling with Jump Processes
  Chapman & Hall/CRC, Boca Raton, FL., 2004

UNIVERSITY OF CALGARY

References

- William Margrabe (1978)
  The Value of an Option to Exchange One Asset for Another, The Journal of Finance, VOL. XXXIII, NO. I
- S. Huang and M. Hung (2005)
  Pricing Foreign Equity Options under Lévy processes, Journal of Futures Markets, 25(10),pp. 917-944
- Black, F. and Scholes, M. (1973)
  The Pricing of Options and Corporate Liability, Journal of Political Economy, v. 81, pp. 637-659.
- Robert G. Tompkins and José C. Wong (2001)
  Exotic Options, European Banking Study, pp: 973-982

Thanks for Coming!