Games with Exhaustible Resources by Chris Harris, Sam Howison, Ronnie Sircar

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Game Theory

- Game theory is a theory about strategic interactions.
- In most instances, no single party, or player, can determine the outcome single-handedly;
- The outcome usually depends on several or all participants' behaviour.

Cournot Game

- A Cournot game is a game between two firms. Both produce a certain good, no other firms do.
- The price they receive is a decreasing function of the total quantity of the product the firms produce.
- That function is known to both firms.
- Each chooses a quantity to produce without knowing how much the other will produce

Cournot Doupoly

- A Cournot duopoly is a pair of firms who split a market, modeled as in the Cournot game.

Differential Games

- Differential games usually consists of two actors, with conflicting goals where the dynamics of the actors are modeled by systems of differential equations.

Zero-Sum Games

- Zero-sum property, is a single performance criterion which one player tries to minimize and the other tries to maximize.

Nonzero-Sum Games

- Nonzero-sum games, are the games where the players wish to minimize different performance criteria.
- This means if we consider players I and II in the nonzero-sum game, II (I) is more interested in maximising his own profit than minimising the I's (II) profit.

What We Are Interested to Study

- Here we will study N-player Cournot games in an oligopoly where firms choose production quantities.
- These games are called nonzero-sum differential games.

  (In game theory the oligopoly game in the market is not a zero-sum game, which means somebody's gain is not necessarily somebody else's loss.)

What We Are Interested to Study

- There is a possibility for the value functions of these games to be characterized by system of nonlinear Hamilton-Jacobi partial differential equations.
- For resources with finite supply, like oil, exhaustibility acts as PDEs boundary conditions.

What We Are Interested to Study

- We analyze those sort of problems with alternative, but expensive technology such as solar power for energy production, and give an asymptotic approximation in the limit of small exhaustibility.

Static Cournot Game

- First we analyze the static game which is a one-period (stage) game (before moving to the dynamic game with exhaustibility).

Static Cournot Game

- Price (inverse-demand) Function:
  $P : (0, \infty) \rightarrow \mathbb{R}$
  (This gives market price as a function of quantity produced)
- $N \geq 1$ is the number of players,
- $q_i \in [0, \infty)$: the quantity that each player i chooses to produce at unit cost of production $a_i \geq 0$,
- Market price is determined from the total production:
  P(Q) where $Q = \sum_{j=1}^{N} q_j$.

Static Cournot Game

The profit of player i is equal to:

$$\pi(q_i, Q_{-i}, a_i) = \begin{cases} q_i(P(Q_{-i} + q_i) - a_i) & q_i > 0 \\ 0 \quad q_i = 0 \end{cases} \tag{1}$$

where $Q_{-i} = \sum_{j \neq i} q_j$ is total production by players other than i. ($P(0^+)$ is called Choke price and $P(0^+) = +\infty$)

Static Cournot Game

- Each player's aim is to maximize his own profit, with respect to the quantities produced by the other players.
- Thus we rewite the previous term as following definition:
- Definition: A Nash equilibrium is a vector $q^* = (q_1^*, q_2^*, ..., q_N^*) \in [0, \infty)^N$such that, for all i,
  $\pi(q_i^*, Q_{-i}^*, a_i) = \max_{q_i \in [0,\infty)} \pi(q_i, Q_{-i}^*, a_i)$
  where $Q_{-i}^* = \sum_{j \neq i} q_j^*$
- If $q_i^* > 0$ for all i, then q^* is called an interior Nash equilibrium.

Nash Equilibrium Existence and Uniqueness

We want to show that, under which conditions on the price function P and the cost vector $a = (a_1, a_2, , ..., a_N)$, a Nash equilibrium exists and is unique.

Nash Equilibrium Existence and Uniqueness

- Assumption on P:
  The price P is twice continuously differentiable, with $P' < 0$ everywhere on $(0, \infty)$; and there exists $\eta \in (0, \infty)$ such that $P(\eta) = 0$.
- We order the firms by their costs and assume they are strictly less than the Choke price $P(0^+)$,i.e.:

$$0 \leqslant a_1 \leqslant a_2 \leqslant ... \leqslant a_N < P(0^+)$$

Nash Equilibrium Existence and Uniqueness

- The behaviour of P is best characterized in terms of the relative prudence of P:
  $\rho(Q) = \frac{-QP''(Q)}{P'(Q)}$

  (A consumer is 'prudent', if he or she save more when faced with riskier future income.)

- We also define:
  $\bar{\rho} = \sup_{Q \in (0,\infty)} \rho(Q)$

General Price Function

- Suppose q^* is an interior Nash equilibrium.
- Then $\forall i \in \{1, 2, 3, ..., N\}$, q_i^* must satisfy the following first order condition:
  $0 = \frac{\partial \pi}{\partial q_i}(q_i^*, Q_{-i}^*, a_i) =$
  $q_i^* P'(Q_{-i}^* + q_i^*) + P(Q_{-i}^* + q_i^*) - a_i$
- So a Candidate Nash equilibrium is :
  $q_i^* = \frac{P(Q^*) - a_i}{-p'(Q^*)}$

General Price Function

- Summing over i:
  $0 = Q^* P'(Q^*) + NP(Q^*) - A_N,$

  where
  $Q^* = Q^*_{-i} + q^*_i = \sum_{i=1}^{N} q^*_i$ is total production,
  $A_N = \sum_{i=1}^{N} a_i$ is the sum of the unit costs,
  and $f_N(Q) = A_N$.

General Price Function

Lemma:

Fix $n \in \{1, 2, ..., N\}$, and suppose that $\bar{\rho} < n + 1$. Then there is a unique $Q_n^* \in (0, \eta)$ such that $f_n(Q_n^*) = A_n$.

- Thus $\forall n > \max(0, \bar{\rho} - 1)$, we have following n-player candidate Nash equilibrium:

$$q_i^* = \begin{cases} \frac{P(Q^*) - a_i}{-p'(Q^*)} & 1 \leqslant i \leqslant n, \\ 0 & n+1 \leqslant i \leqslant N \end{cases} \qquad (2)$$

where Q_n^* is the unique solution of $f_n(Q) = A_n$.

Failure of the candidate equilibrium as a Nash equilibrium

- This candidate equilibrium can fail to be a Nash equilibrium of the game as a whole in one of following three ways:
  1. It may happen that $q_i^* < 0$ for some $1 \leqslant i \leqslant n$;
  2. It may happen that $a_i < P(Q_n^*)$ for some $n+1 \leqslant i \leqslant N$;
  3. It may happen that for some $1 \leqslant i \leqslant n$, q_i^* is not a global maximum of $\pi(\cdot, Q_{-i}, a_i)$.

The third case can be eliminated by a hypothesis on P:

Lemma:

Suppose that $\bar{\rho} < 2$ and $Q_{-i} \geq 0$. Then $g(q_i) := \pi(q_i, Q_{-i}, a_i)$ has a unique global maximum, which is attained in $[0, \eta)$.

Proposition:
Suppose that $\bar{\rho} < 2$. Then there is a unique Nash equilibrium.

- When a Nash equilibrium exists and is unique, we denote the equilibrium production of player i as a function of the vector of costs
$a = (a_1, a_2, a_3, ..., a_N)$
by $q_i^*(a)$ and the equilibrium profit of player i by:

$$G_i(a) = \pi(q_i^*(a), Q_{-i}^*(a), a_i),$$

where $Q_{-i}^*(a) = \sum_{i \neq j} q_j^*(a)$

- Corollary:
  Suppose that $\bar{\rho} < 2$. Then the unique Nash equilibrium can be constructed as follows. Let $\bar{Q}^* = \max\{Q_n^* | 1 \leqslant n \leqslant N\}$. Then the unique Nash equilibrium quantities are given by:

  $$q_i^*(a) = \max\{\frac{P(\bar{Q}^*) - a_i}{-p'(\bar{Q}^*)}, 0\}, \ 1 \leqslant i \leqslant N$$

  and the corresponding profits are:

  $$G_i(a) = q_i^*(a)(P(\bar{Q}^*) - a_i), \ 1 \leqslant i \leqslant N$$

  and the number of active players in the unique equilibrium is $m = \min\{n | Q_n^* = \bar{Q}^*\}$

Cournot Prudence Price Curves

- We define $n_\rho = \max(1, \lfloor \rho \rfloor)$,

  where $\lfloor \rho \rfloor$ denotes the largets integer less than or equal to ρ.

Cournot Prudence Price Curves

- Lemma: There is no Nash equilibrium for $n < \lfloor \rho \rfloor$. For $n \geq n_\rho$, there is a unique solution $Q_n^* \in (0, \eta)$ to $f_n(Q) = A_n$, $\forall 0 \leqslant A_n < nP(0^+)$.

- Lemma: Suppose that $2 \leqslant \rho < N + 1$, and $Q_{-i} > 0$. Then $g(q_i) := \pi(q_i, Q_{-i}, a_i)$ has a unique global maximum, which is attained in $[0, \eta)$.

Cournot Prudence Price Curves

- Proposition: Suppose that$\rho < N + 1$. Then there is a unique Nash equilibrium given as follows:

  $q_i^*(a) = (\frac{\bar{Q}}{\eta})^\rho \max\{\bar{P} - a_i, 0\}$, $1 \leqslant i \leqslant N$,

  where

  $\bar{p} = \min\{P_n | n_\rho \leqslant n \leqslant N\}$,

  $P_n = \frac{A_n + \eta}{n+1-\rho}$, $n_\rho \leqslant n \leqslant N$

  and

$$\bar{Q} = \begin{cases} \eta(1 - (1-\rho)\frac{\bar{p}}{\eta})^{\frac{1}{1-\rho}} & \rho \neq 1 \\ \eta exp(\frac{-\bar{P}}{\eta}) & \rho = 1 \end{cases} \qquad (3)$$

Cournot Prudence Price Curves

- The corresponding profits are:
  $G_i(a) = q_i^*(a)(\bar{P} - a_i)$

  and the number of active players in the unique equilibrium is :

  $$m = \min\{n | n_\rho \leqslant n \leqslant N, P_n = \bar{P}\}.$$

Differential Game and Exhaustibility

- We now introduce the dynamic Cournot game under exhaustibility constraints.
- Each player i has reserves of a traditional and cheap-to-produce resource (for instance oil, by extraction),denoted by $x_i(t)$ at time $t \geq 0$.

Differential Game and Exhaustibility

- For simplicity we take the cost of production from this source to be zero, but reserves are finite(exhaustible).
- There is also an alternative source that is inexhaustible but expensive-to-produce (like solar power in the energy), with constant unit cost of production $c \in [0, P(0^+))$.

Differential Game and Exhaustibility

- Player i chooses a dynamic production rate $\bar{q}_i$. $\bar{q}_i = \bar{q}_i(x(t))$, where $x(t) = (x_1(t), ..., x_N(t))$.
- As long as $x_i > 0$, player i has the choice between producing from the cheap or expensive source.
- After x_i hits zero, he can only produce from the costlier alternative, which never runs out.

Traditional reserve resource depletion

- The reserves of the traditional resource deplete according to:

$$\frac{dx_i}{dt} = -\bar{q}_i(x(t)),\ x_i > 0.$$

Dynamic Cournot Competition

- Given initial reserves $x_i(0) \geq 0$, player i wants to maximize his discounted lifetime pofit i.e.:

  $\int_0^\infty e^{-rt}\pi(\bar{q}_i(x(t)), \bar{Q}_{-i}(x(t)), c1_{\{x_i(t)=0\}})dt,$

  where:

  1 denotes the indicator function,

  $r > 0$ is a discount rate,

  π (profit function) was defined before,

  and $\bar{Q}_{-i} = \sum_{i \neq j} \bar{q}_j$.
- Note that the cost of production rises from zero to c when reserves x_i run out.

Dynamic Cournot Competition

- With the notation $\bar{Q}^*_{-i} = \sum_{i \neq j} \bar{q}^*_j$:

  $\int_0^\infty e^{-rt}\pi(\bar{q}_i{}^*(x(t)), \bar{Q}^*_{-i}(x(t)), c1_{\{x_i(t)=0\}})dt \geq \int_0^\infty e^{-rt}\pi(\bar{q}_i(x(t)), \bar{Q}^*_{-i}(x(t)), c1_{\{x_i(t)=0\}})dt,$

  for any strategy $\bar{q}_i$ of player i,and $\forall x(0) \in [0, \infty)^N$.

- $\bar{q}_i{}^*$is the best response when all the other players play their equilibrium strategies.

Dynamic Cournot Competition

- Give an informal motivation for the dynamic programming PDEs we shall use to construct Nash equilibria for these problems.
- First:

  consider any strategy $\{\bar{q}_j(x(t))|t \geq 0, 1 \leqslant j \leqslant N\}$, and the profits starting at time $s \geq 0$:

  $$v_i^{\bar{q}}(x(s)) = \int_s^{\infty} e^{-r(t-s)} \pi(\bar{q}_i(x(t)), \bar{Q}_{-i}(x(t)), c1_{\{x_i(t)=0\}}) dt$$
  $$, 1 \leqslant i \leqslant N$$

  ($v_i^{\bar{q}}(x(s))$ is called value function)

Dynamic Cournot Competition

- Putting $v_i = v_i^{\bar{q}^*}$ and using the equilibrium policies:

$$v_i(x) = \int_0^\infty e^{-rt}\pi(\bar{q}_i{}^*(x(t)), \bar{Q}^*_{-i}(x(t)), c1_{\{x_i(t)=0\}})dt$$

- We have

$$\max_{q_i \geq 0}[\pi(q_i, \bar{Q}^*_{-i}(x), \frac{\partial v_i}{\partial x_i})] - \sum_{i \neq j} \bar{q}_j{}^*(x)\frac{\partial v_i}{\partial x_j} - rv_i = 0 \tag{4}$$

$$a_i = \frac{\partial v_i}{\partial x_i}(x),\ i = 1, 2, 3, ..., N.$$

Dynamic Cournot Competition

- Then we rewrite equation (4) as

$$G_i(Dv) - \sum_{j \neq i} q_j^*(Dv) \frac{\partial v_i}{\partial x_j} - rv_i = 0, i = 1, ..., N \tag{5}$$

  where we define
  $Dv = diag(\nabla v) = (\frac{\partial v_1}{\partial x_1}, ..., \frac{\partial v_N}{\partial x_N})$

- Recall $G_i(a) = q_i^*(a)(P(Q^*) - a_i)$ is equilibrium profit function of the state game.

Dynamic Cournot Competition

- The equilibrium production rates of the exhaustible resource at time t are given by $\bar{q}_i{}^*(x(t)) = q_i^*(Dv(x(t)))$

Exhaustibility

- Consider the case $x_i = 0$, when player i has exhausted his supply. Then we have $\frac{dx_i}{dt} = 0$.
- The HJ equation for the $v_i(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_N)$ becomes

  $\max_{q_i \geq 0} \pi(q_i, \bar{Q}^*_{-i}(x), c) - \sum_{j \neq i} q^*_j(Dv) \frac{\partial v_i}{\partial x_j} - rv_i = 0$

Effect of Small-costs on the Static Game

- Static Game Small Cost Perturbation (Effect of Small-costs on the Static Game):

$$G_i(a) = q_i^*(a)(P(Q^*) - a_i)$$

Effect of Small-costs on the Static Game

- Small costs Taylor expansion gives:

  $G_i(a) \approx G_i(0) + Aa_i + B\sum_{j\neq i} a_j,$

  where : $A = \frac{\partial G_i}{\partial a_i}(0)$ $B = \frac{\partial G_i}{\partial a_j}(0)$ $j \neq i$
- Similarly:

  $q_i^*(a) \approx \gamma + \lambda a_i + \mu \sum_{j\neq i} a_j$

  where $\gamma = q_i^*(0)$ $\lambda = \frac{\partial q_i^*}{\partial a_i}(0)$ $\mu = \frac{\partial q_i^*}{\partial a_j}(0)$ $j \neq i$
- Also we define the constant of relative prudence at the zero cost equilibrium solution by:

  $\rho_0 = -N_\gamma \frac{P''(N_\gamma)}{P'(N_\gamma)}$

Effect of Small-costs on the Static Game

- Proposition: We assume $\rho_0 < (N+1)$. The perturbation coefficients (A, B, λ, μ) can be expressed as:

$$A = -\gamma[\frac{2N - (2 - N^{-1})]\rho_0}{(N+1) - \rho_0}] \quad (6)$$

$$B = \gamma[\frac{2 - N^{-1}\rho_0}{(N+1) - \rho_0}] \quad (7)$$

Effect of Small-costs on the Static Game

$$\lambda = \frac{1}{P'(N_\gamma)}[\frac{N-(1-N^{-1})\rho_0}{(N+1)-\rho_0}] \tag{8}$$

$$\mu = -\frac{1}{P'(N)_\gamma}[\frac{1-N^{-1}\rho_0}{(N+1)-\rho_0}] \tag{9}$$

Effect of Small-costs on the Static Game

- Since $\rho_0 < (N+1)$ and $N \geq 2$ thus:
  $B > 0$ and $\lambda < 0$.
  $A \leqslant 0$ for $\rho_0 \leqslant \frac{2N^2}{2N-1}$, and $A > 0$ otherwise.
  $\mu \geq 0$ for $\rho_0 \leqslant N$ and $\mu < 0$ otherwise.

Effect of Small-costs on the Static Game

- $B = \frac{\partial G_i}{\partial a_j}(0) > 0$: Any player i's profits increase when any other player's cost a_j is increased from zero, and he also increases his production for $\rho_0 \leqslant N.(\mu = \frac{\partial q_i^*}{\partial a_j}(0) \geq 0)$.

- $\lambda = \frac{\partial q_i^*}{\partial a_i}(0) < 0$: When player i's cost a_i is increased from zero, he decreases his production, but his profit may increase or decrease depending on ρ_0 ($A = \frac{\partial G_i}{\partial a_i}(0) \leqslant 0$ for $\rho_0 \leqslant \frac{2N^2}{2N-1}$, and $A > 0$ otherwise).

To be Continued...

References

Haris, C., Howison, S., & Sircar, R. 2010. Games with Exhaustible Resources.

Starr, A.W.,& Ho, Y.C. 1969. Nonzero-Sum Differential Games,*Journal of Optimizataion Theory and Applications*, **3**(3).

Weibull, J. W. 2010. Lecture Notes in Game Theory.