# Numerical Solution of Stochastic Differential Equations with Jumps in Finance

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## Jump-Diffusion Multi-Factor Models

Björk, Kabanov & Runggaldier (1997)
Øksendal & Sulem (2005)

- Markovian
- explicit transition densities in special cases
- benchmark framework
- discrete time approximations
- suitable for simulation
- Markov chain approximations

## Pathwise Approximations:

- scenario simulation of entire markets
- testing statistical techniques on simulated trajectories
- filtering hidden state variables
  Pl. & Runggaldier (2005, 2007)
- hedge simulation
- dynamic financial analysis
- extreme value simulation
- stress testing

$\Longrightarrow$ higher order strong schemes
predictor-corrector methods

## Probability Approximations:

- derivative prices
- sensitivities
- expected utilities
- portfolio selection
- risk measures
- long term risk management

$\Longrightarrow$ Monte Carlo simulation, higher order weak schemes,
predictor-corrector,
variance reduction, Quasi Monte Carlo,
or Markov chain approximations, lattice methods

# Essential Requirements:

- parsimonious models
- respect no-arbitrage in discrete time approximation
- numerically stable methods
- efficient methods for high-dimensional models
- higher order schemes, predictor-corrector

# Continuous and Event Driven Risk

- **Wiener processes** $W^k,\ k \in \{1, 2, \ldots, m\}$

- **counting processes** $p^k$

  intensity $h^k$

  jump martingale $q^k$

$$dW_t^{m+k} = dq_t^k = \left(dp_t^k - h_t^k\, dt\right) \left(h_t^k\right)^{-\frac{1}{2}}$$

$k \in \{1, 2, \ldots, d-m\}$

$$W_t = (W_t^1, \ldots, W_t^m, q_t^1, \ldots, q_t^{d-m})^\top$$

## Primary Security Accounts

$$dS_t^j = S_{t-}^j \left( a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right)$$

**Assumption 1**

$$b_t^{j,k} \geq -\sqrt{h_t^{k-m}}$$

$k \in \{m+1, \ldots, d\}$.

**Assumption 2**

*Generalized volatility matrix* $b_t = [b_t^{j,k}]_{j,k=1}^d$ invertible.

- **market price of risk**

$$\theta_t = (\theta_t^1, \ldots, \theta_t^d)^\top = b_t^{-1} [a_t - r_t \, \mathbf{1}]$$

- **primary security account**

$$dS_t^j = S_{t-}^j \left( r_t \, dt + \sum_{k=1}^{d} b_t^{j,k} \, (\theta_t^k \, dt + dW_t^k) \right)$$

- **portfolio**

$$dS_t^\delta = \sum_{j=0}^{d} \delta_t^j \, dS_t^j$$

- **fraction**

$$\pi^j_{\delta,t} = \delta^j_t \, \frac{S^j_t}{S^\delta_t}$$

- **portfolio**

$$dS^\delta_t = S^\delta_{t-} \Big\{ r_t \, dt + \pi^\top_{\delta,t-} \, b_t \, (\theta_t \, dt + dW_t) \Big\}$$

**Assumption 3**

$$\sqrt{h_t^{k-m}} > \theta_t^k$$

- **generalized GOP volatility**

$$c_t^k = \begin{cases} \theta_t^k & \text{for} \quad k \in \{1, 2, \ldots, m\} \\ \frac{\theta_t^k}{1-\theta_t^k \, (h_t^{k-m})^{-\frac{1}{2}}} & \text{for} \quad k \in \{m+1, \ldots, d\} \end{cases}$$

- **GOP fractions**

$$\pi_{\delta_*,t} = (\pi_{\delta_*,t}^1, \ldots, \pi_{\delta_*,t}^d)^\top = \left(c_t^\top \, b_t^{-1}\right)^\top$$

- **Growth Optimal Portfolio**

$$dS_t^{\delta_*} = S_{t-}^{\delta_*}\Big(r_t\, dt + c_t^\top\, (\theta_t\, dt + dW_t)\Big)$$

- **optimal growth rate**

$$g_t^{\delta_*} \;=\; r_t + \frac{1}{2}\sum_{k=1}^{m}(\theta_t^k)^2$$

$$- \sum_{k=m+1}^{d} h_t^{k-m}\left(\ln\left(1 + \frac{\theta_t^k}{\sqrt{h_t^{k-m}} - \theta_t^k}\right) + \frac{\theta_t^k}{\sqrt{h_t^{k-m}}}\right)$$

- **benchmarked portfolio**

$$\hat{S}_t^{\delta} = \frac{S_t^{\delta}}{S_t^{\delta_*}}$$

**Theorem 4** *Any nonnegative benchmarked portfolio $\hat{S}^{\delta}$ is an $(\underline{\mathcal{A}}, P)$-supermartingale.*

$\Longrightarrow$ no strong arbitrage

but there may exist:

free lunch with vanishing risk (Delbaen & Schachermayer (2006))

free snacks or cheap thrills (**?**))

# Multi-Factor Model

model mainly:

- **benchmarked primary security accounts**

$$\hat{S}_t^j = \frac{S_t^j}{S_t^{\delta_*}}$$

$j \in \{0, 1, \ldots, d\}$

supermartingales, often SDE driftless,
local martingales, sometimes martingales

savings account

$$S_t^0 = \exp\left\{\int_0^t r_s\, ds\right\}$$

$\Longrightarrow$ GOP

$$S_t^{\delta_*} = \frac{S_t^0}{\hat{S}_t^0}$$

$\Longrightarrow$ stock

$$S_t^j = \hat{S}_t^j\, S_t^{\delta_*}$$

additionally dividend rates

foreign interest rates

# Example

## Black-Scholes Type Market

$$d\hat{S}_t^j = -\hat{S}_{t-}^j \sum_{k=1}^{d} \sigma_t^{j,k}\, dW_t^k$$

$h_t^j,\ \sigma_t^{j,k},\ r_t$

# Examples

- **Merton jump-diffusion model**

$$dX_t = X_{t-}\,(\mu\,dt + \sigma\,dW_t + dp_t)\,,$$

$$\Downarrow$$

$$X_t = X_0\,e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t}\,\prod_{i=1}^{N_t}\xi_i$$

- **Bates model**

$$dS_t = S_{t-}\left(\alpha\,dt + \sqrt{V_t}\,dW_t^S + dp_t\right)$$

$$dV_t = \xi(\eta - V_t)\,dt + \theta\sqrt{V_t}\,dW_t^V$$

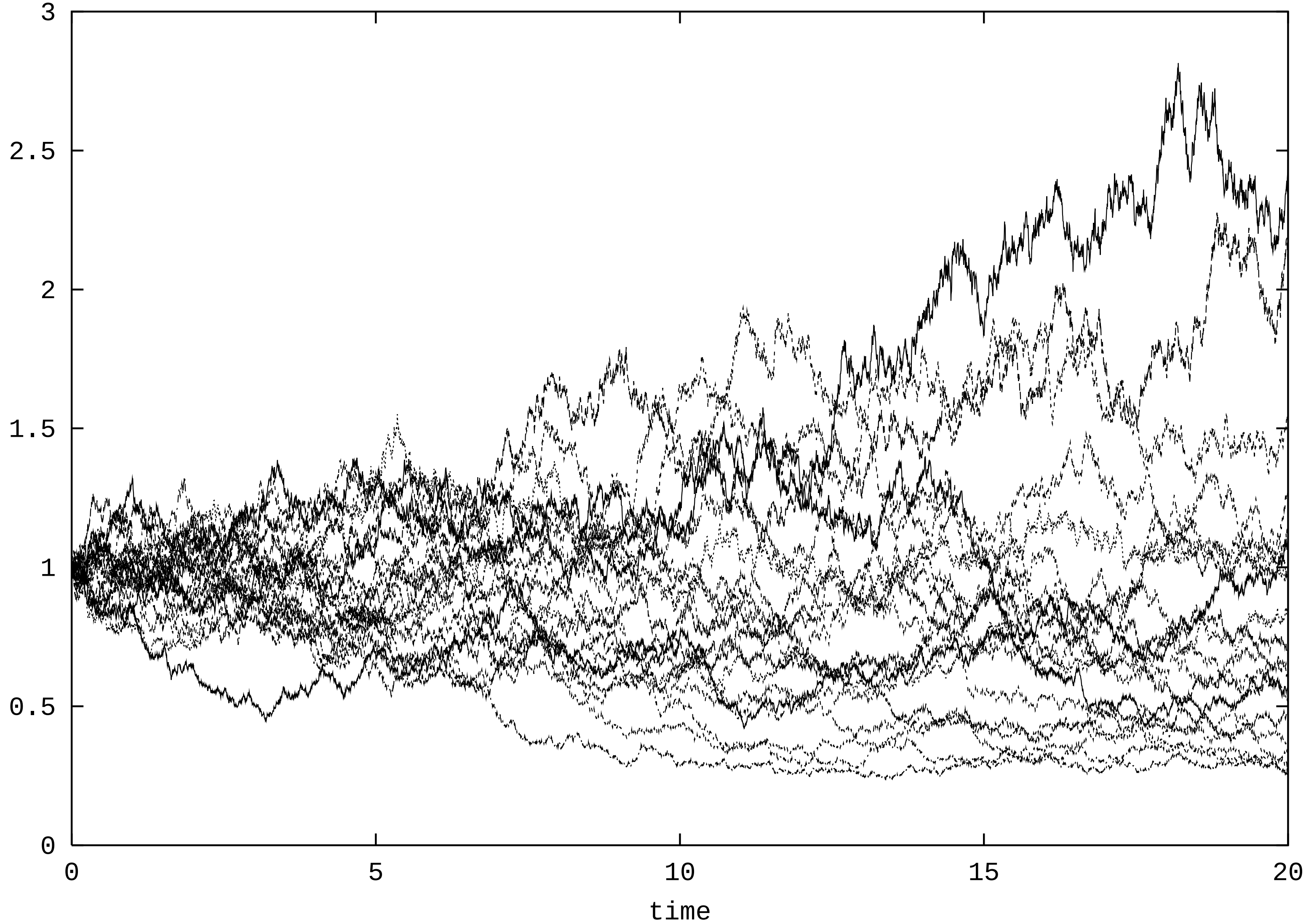



Figure 1: Simulated benchmarked primary security accounts.

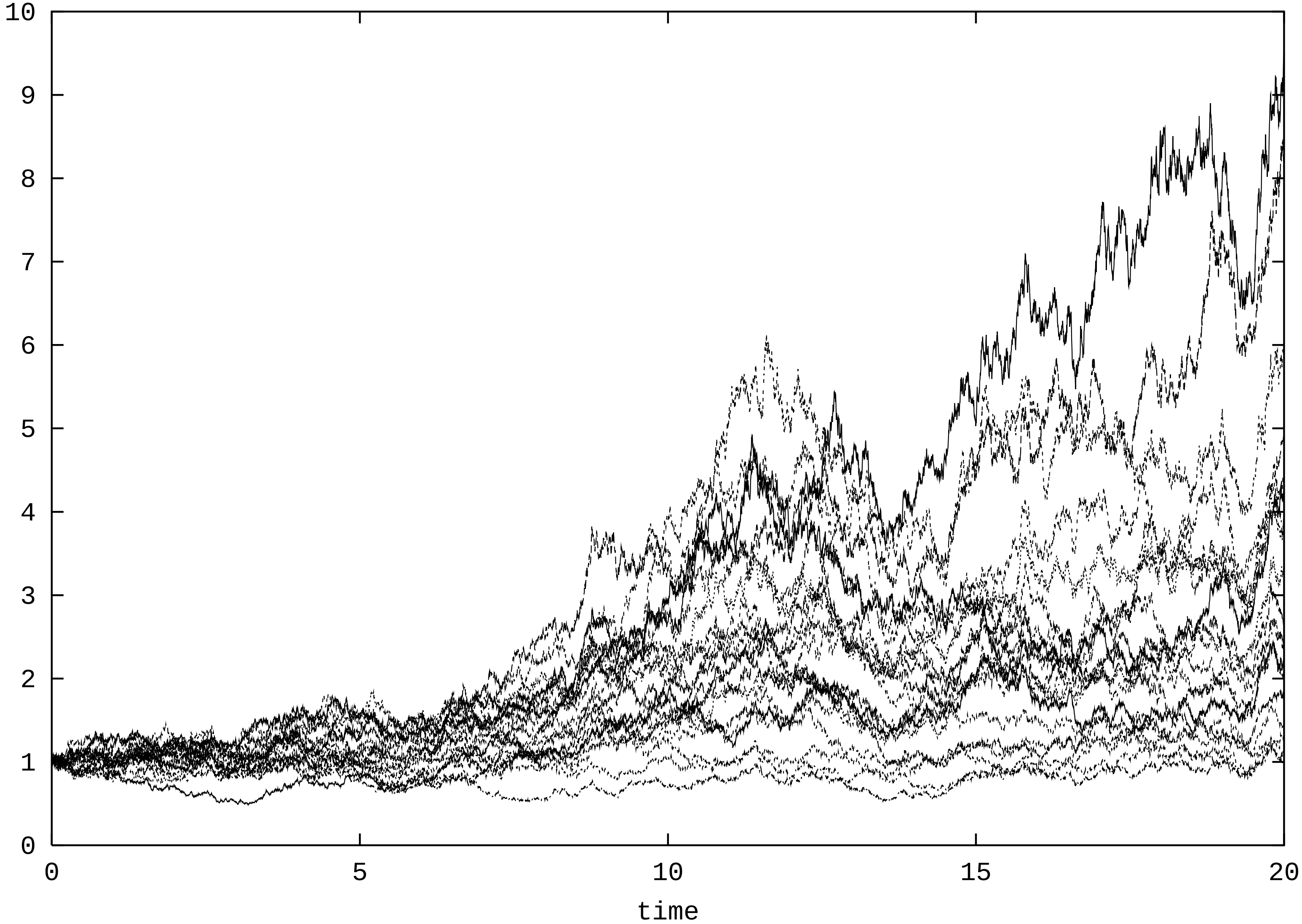



Figure 2: Simulated primary security accounts.

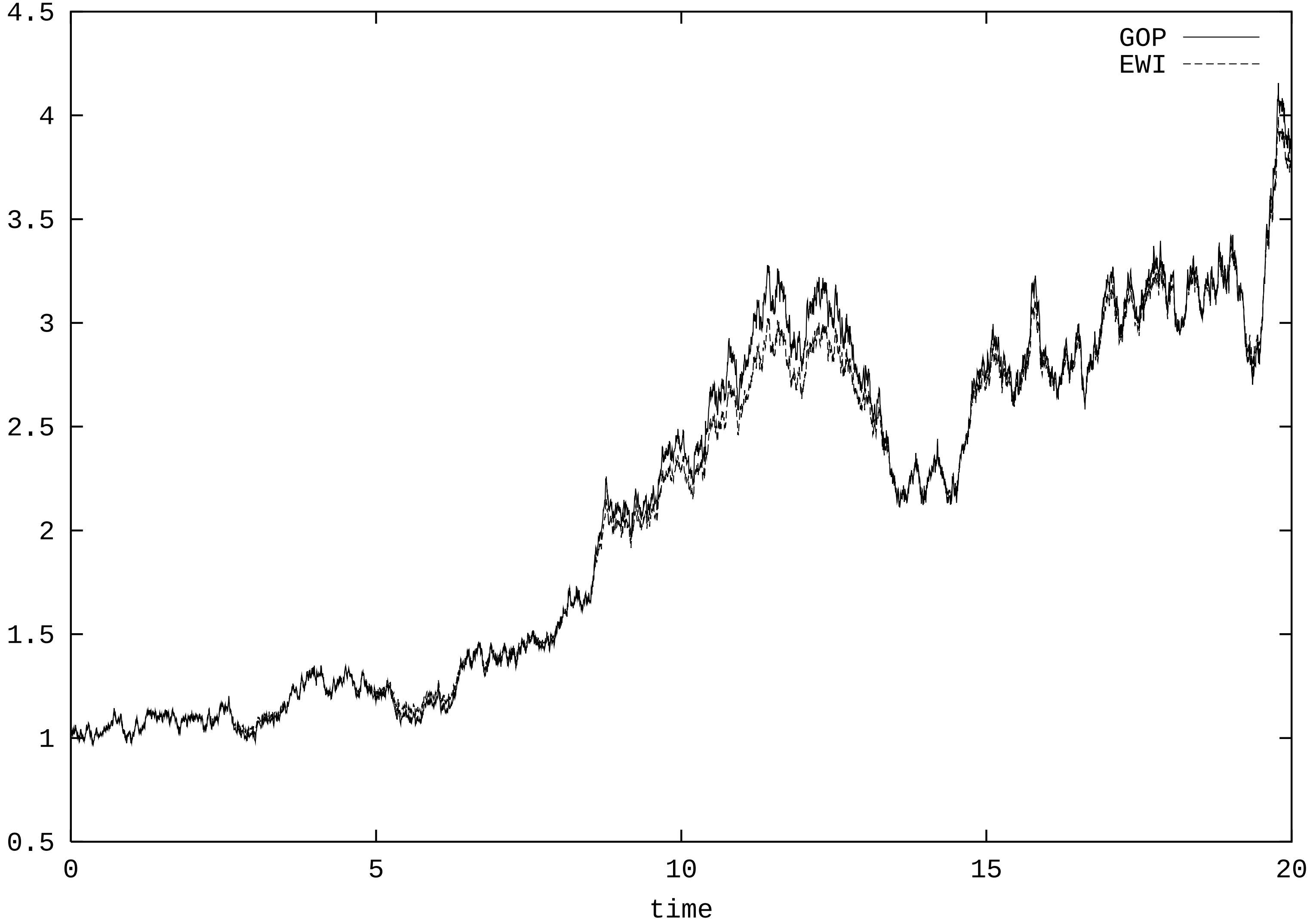



Figure 3: Simulated GOP and EWI for $d = 50$.

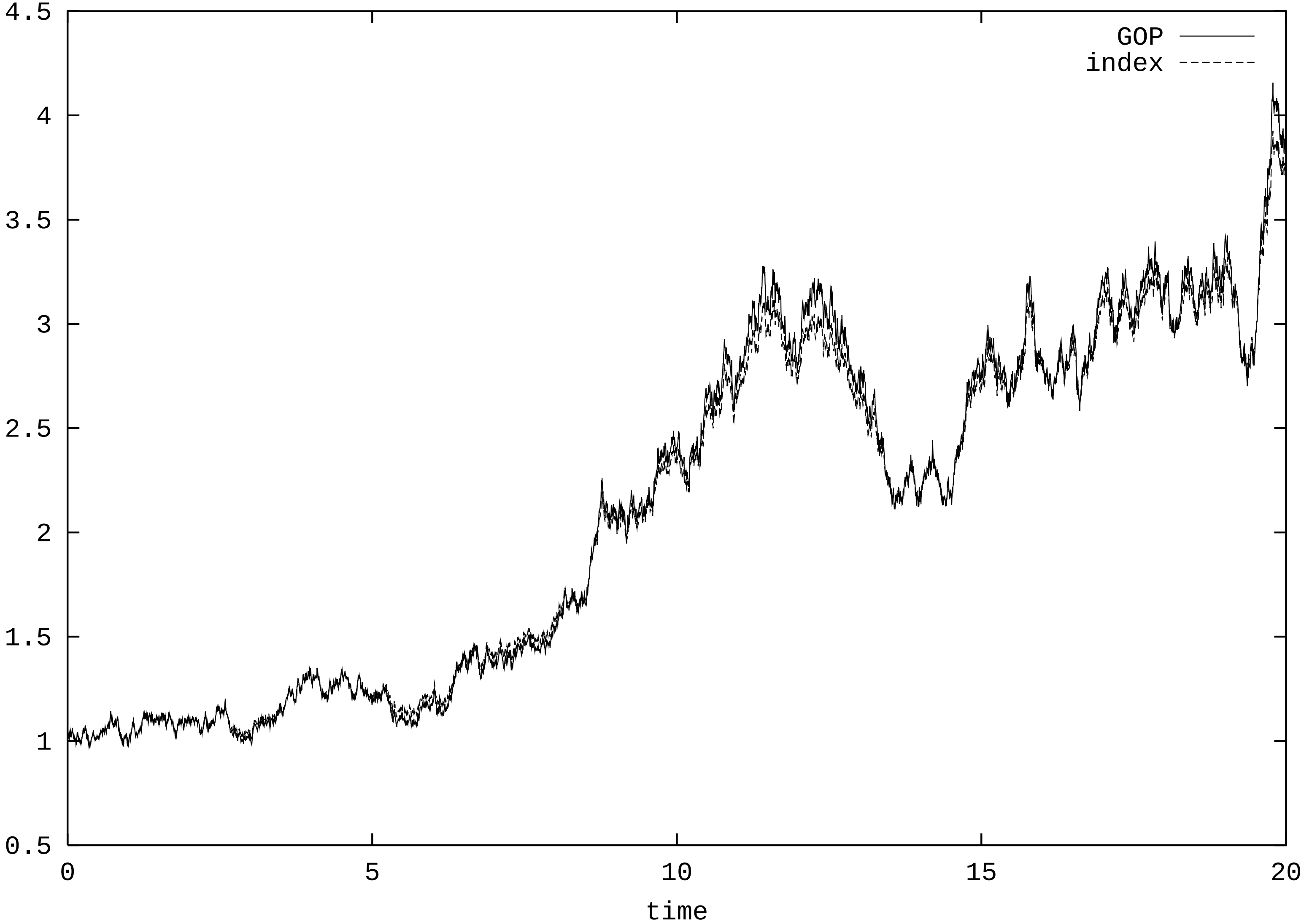



Figure 4: Simulated accumulation index and GOP.

## Diversification

- **diversified portfolios**

$$\left|\pi^j_{\delta,t}\right| \leq \frac{K_2}{d^{\frac{1}{2}+K_1}}$$

**Theorem 5**

*In a regular market any diversified portfolio is an approximate GOP.*

Pl. (2005)

- robust characterization
- similar to Central Limit Theorem
- model independent

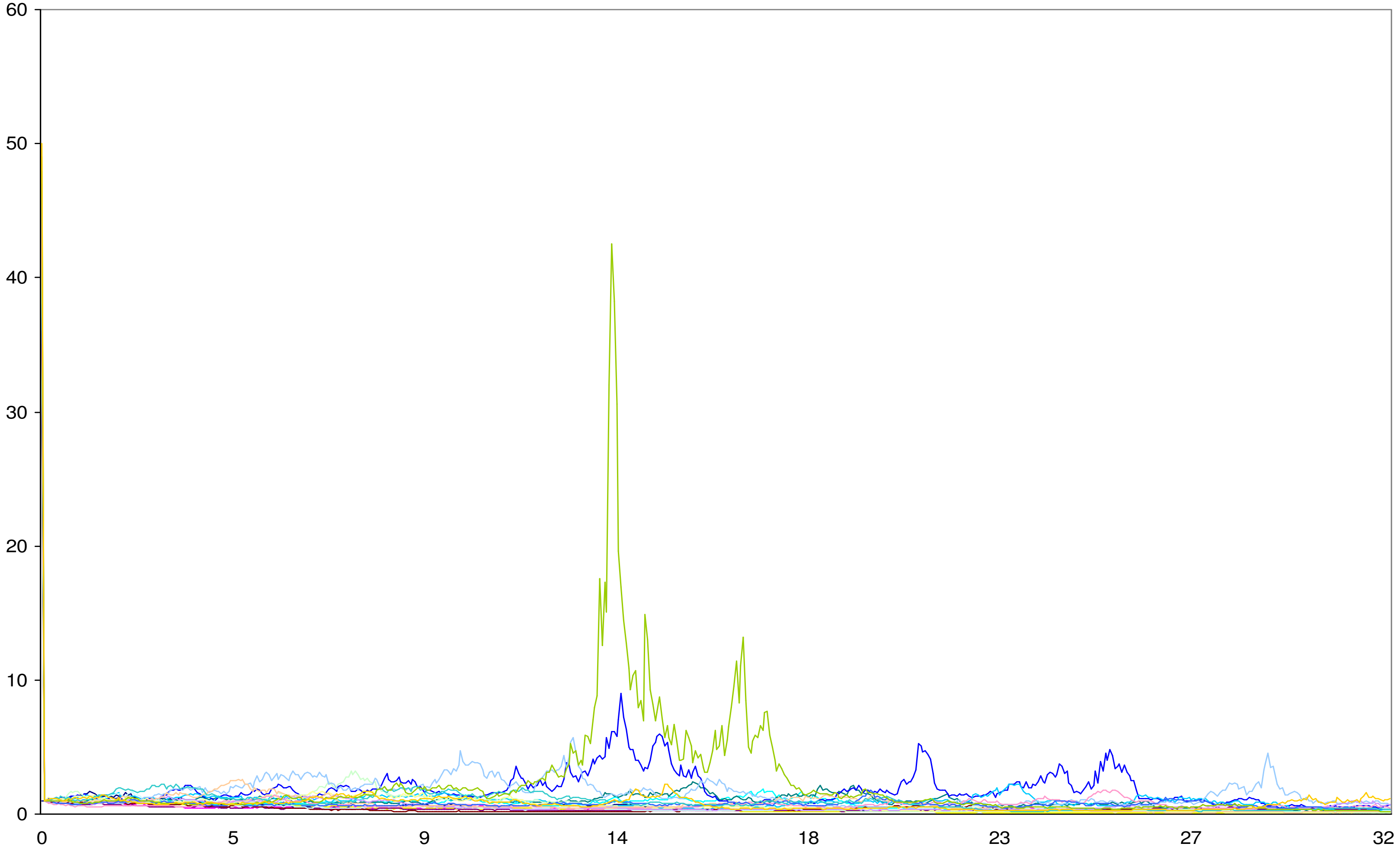



Figure 5: Benchmarked primary security accounts.

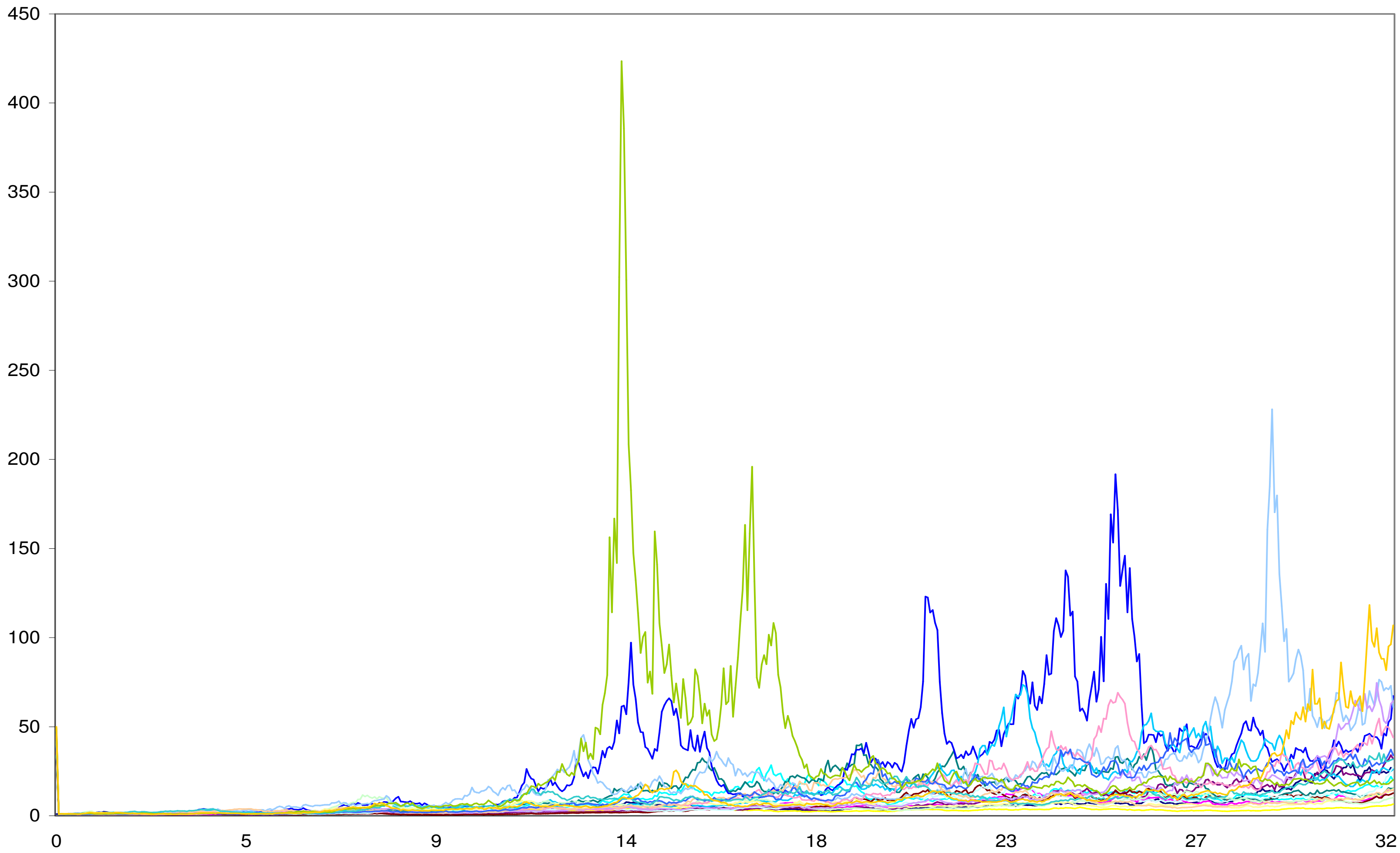



Figure 6: Primary security accounts under the MMM.

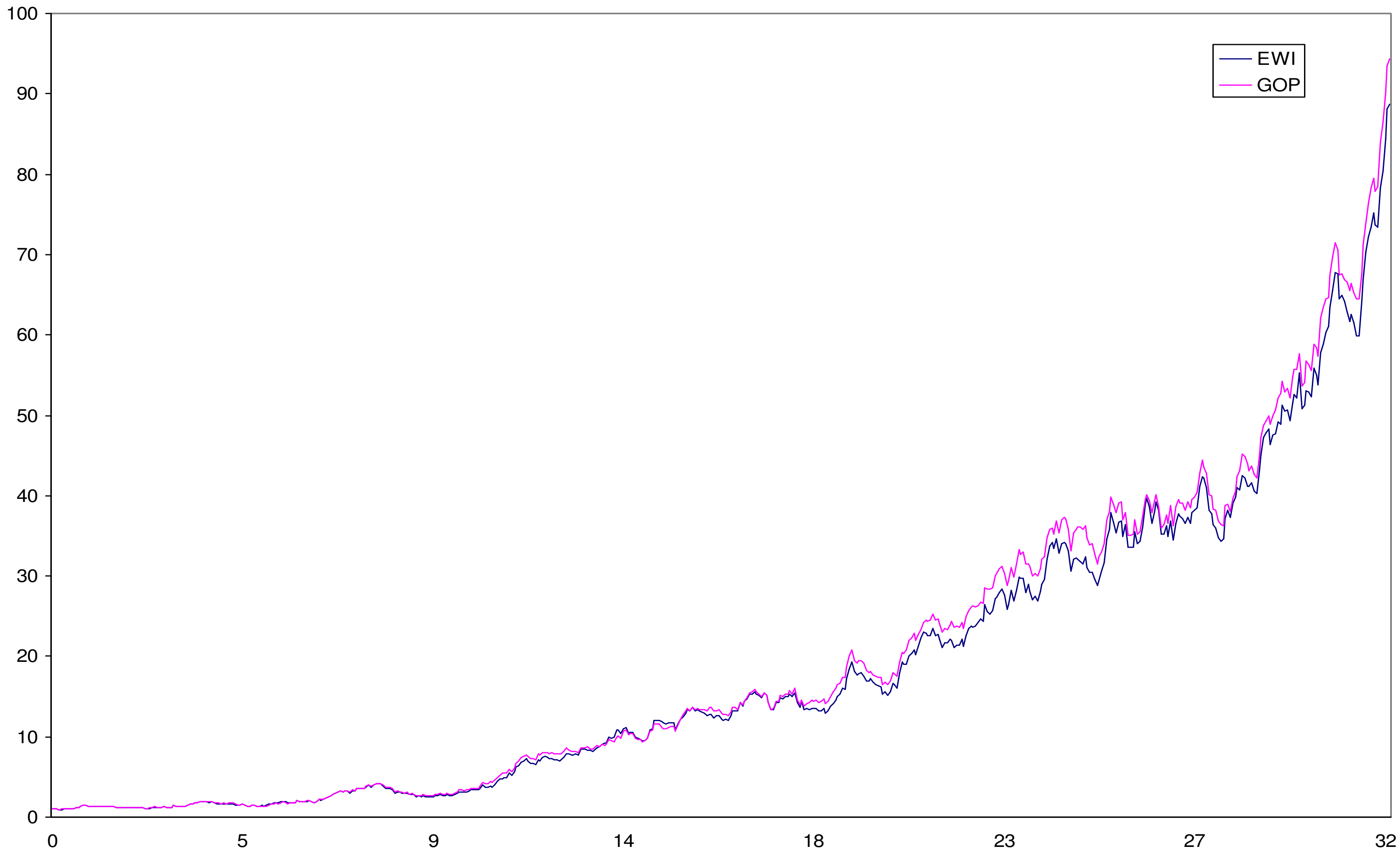



Figure 7: GOP and EWI.

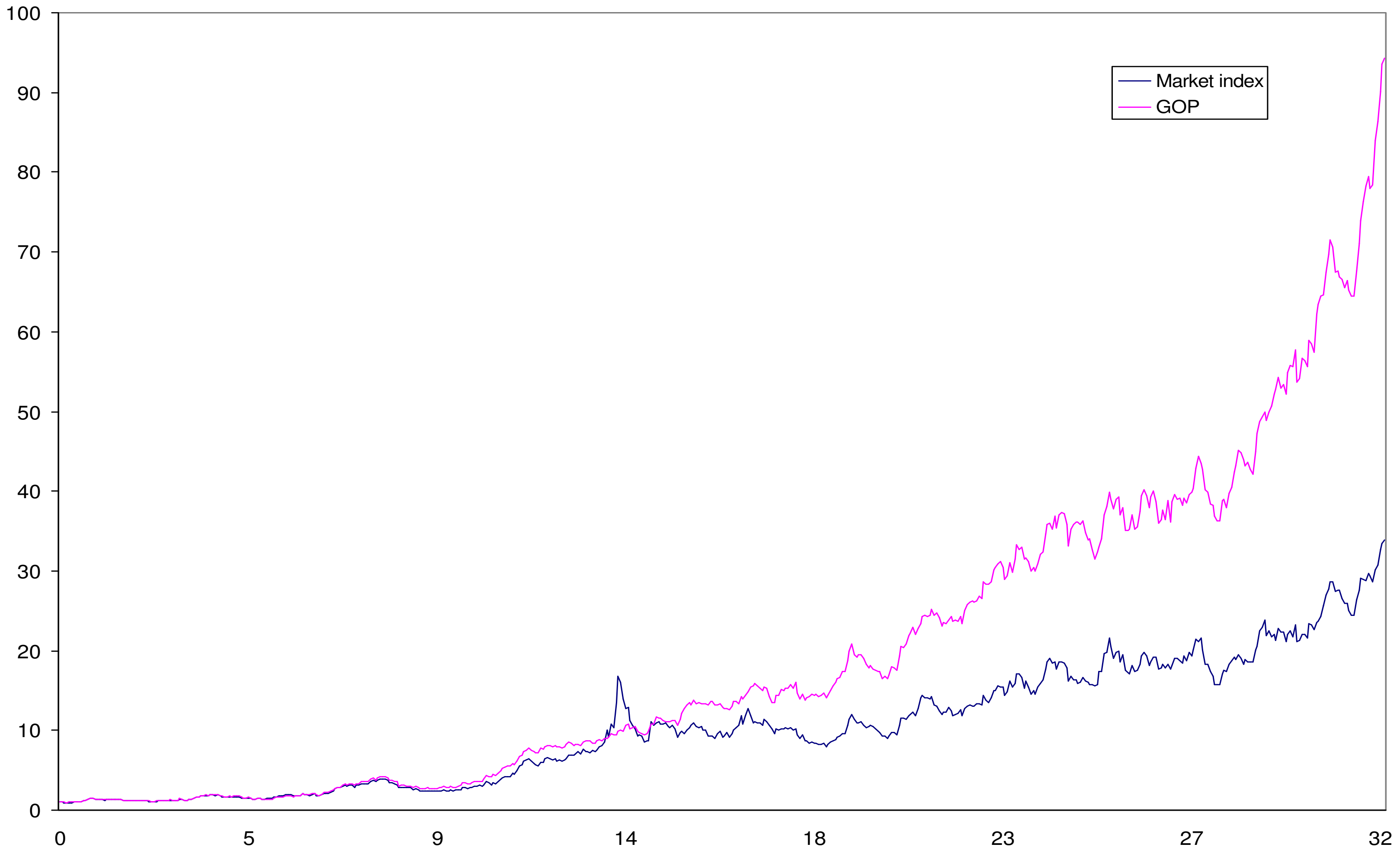



Figure 8: GOP and market index.

- **fair security**

  benchmarked security $(\underline{\mathcal{A}}, P)$-martingale $\Longleftrightarrow$ fair

- **minimal replicating portfolio**

  fair nonnegative portfolio $S^\delta$ with $S^\delta_\tau = H_\tau$

  $\Longrightarrow$ minimal nonnegative replicating portfolio

- **fair pricing formula**

$$V_{H_\tau}(t) = S^{\delta_*}_t \, E\left( \frac{H_\tau}{S^{\delta_*}_\tau} \,\middle|\, \mathcal{A}_t \right)$$

No need for equivalent risk neutral probability measure!

# Fair Hedging

- fair portfolio $\quad S_t^\delta$
- benchmarked fair portfolio

$$\hat{S}_t^\delta = E\left(\frac{H_\tau}{S_\tau^{\delta_*}} \,\middle|\, \mathcal{A}_t\right)$$

- **martingale representation**

$$\frac{H_\tau}{S_\tau^{\delta_*}} = E\left(\frac{H_\tau}{S_\tau^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) + \sum_{k=1}^{d} \int_t^\tau x_{H_\tau}^k(s)\, dW_s^k + M_{H_\tau}(t)$$

$M_{H_\tau}$-$(\underline{\mathcal{A}}, P)$-martingale (pooled)

$$E\left(\left[M_{H_\tau}, W^k\right]_t\right) = 0$$

Föllmer & Schweizer (1991), Pl. & Du (2011)

No need for equivalent risk neutral probability measure!

# Simulation of SDEs with Jumps

- **strong schemes** (paths)
  Taylor
  explicit
  derivative-free
  implicit
  balanced implicit
  predictor-corrector
- **weak schemes** (probabilities)
  Taylor
  simplified
  explicit
  derivative-free
  implicit, predictor-corrector

- **intensity of jump process**
  - regular schemes $\Longrightarrow$ high intensity
  - jump-adapted schemes $\Longrightarrow$ low intensity

# SDE with Jumps

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t + c(t-, X_{t-})\, dp_t$$

$X_0 \in \Re^d$

- $p_t = N_t$: Poisson process, intensity $\lambda < \infty$
- $p_t = \sum_{i=1}^{N_t}(\xi_i - 1)$: compound Poisson, $\xi_i$ i.i.d r.v.
- Poisson random measure

$$\int_{\mathcal{E}} c(t-, X_{t-}, v)\, p_\phi(dv \times dt)$$

- $\{(\tau_i, \xi_i), i = 1, 2, \ldots, N_T\}$

## Numerical Schemes

- time discretization

$$t_n = n\Delta$$

- discrete time approximation

$$Y_{n+1}^{\Delta} = Y_n^{\Delta} + a(Y_n^{\Delta})\Delta + b(Y_n^{\Delta})\Delta W_n + c(Y_n^{\Delta})\Delta p_n$$

## Strong Convergence

- **Applications:** scenario analysis, filtering and hedge simulation

- **strong order** $\gamma$ if

$$\varepsilon_s(\Delta) = \sqrt{E\left(\left|X_T - Y_N^{\Delta}\right|^2\right)} \leq K\,\Delta^{\gamma}$$

## Weak Convergence

- **Applications:** derivative pricing, utilities, risk measures

- **weak order** $\beta$ if

$$\varepsilon_w(\Delta) = |E(g(X_T)) - E(g(Y_N^\Delta))| \leq K\Delta^\beta$$

# Literature on Strong Schemes with Jumps

- Pl (1982), Mikulevicius & Pl (1988)

  $\Longrightarrow \quad \gamma \in \{\mathbf{0.5, 1, \ldots}\}$ Taylor schemes and jump-adapted

- Maghsoodi (1996, 1998) $\quad \Longrightarrow \quad$ strong schemes $\gamma \leq \mathbf{1.5}$

- Jacod & Protter (1998) $\quad \Longrightarrow \quad$ Euler scheme for semimartingales

- Gardoǹ (2004) $\quad \Longrightarrow \quad \gamma \in \{\mathbf{0.5, 1, \ldots}\}$ strong schemes

- Higham & Kloeden (2005) $\quad \Longrightarrow \quad$ implicit Euler scheme

- Bruti-Liberati & Pl (2007) $\quad \Longrightarrow \quad \gamma \in \{\mathbf{0.5, 1, \ldots}\}$
  explicit, implicit, derivative-free, predictor-corrector

# Euler Scheme

- **Euler scheme**

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n + c(Y_n)\Delta p_n$$

where

$\Delta W_n \sim \mathcal{N}(0, \Delta)$ and $\Delta p_n = N_{t_{n+1}} - N_{t_n} \sim Poiss(\lambda\,\Delta)$

- $\gamma = 0.5$

## Strong Taylor Scheme

Wagner-Platen expansion $\quad \Longrightarrow$

$$\begin{aligned} Y_{n+1} &= Y_n + a(Y_n)\Delta + b(Y_n)\Delta W_n + c(Y_n)\Delta p_n + b(Y_n)b'(Y_n)\, I_{(1,1)} \\ &\quad + b(Y_n)\, c'(Y_n)\, I_{(1,-1)} + \{b(Y_n + c(Y_n)) - b(Y_n)\}\, I_{(-1,1)} \\ &\quad + \{c\,(Y_n + c(Y_n)) - c(Y_n)\}\, I_{(-1,-1)} \end{aligned}$$

with

$$I_{(1,1)} = \tfrac{1}{2}\{(\Delta W_n)^2 - \Delta\}, \qquad I_{(-1,-1)} = \frac{1}{2}\{(\Delta p_n)^2 - \Delta p_n\}$$

$$I_{(1,-1)} = \sum_{i=N(t_n)+1}^{N(t_{n+1})} W_{\tau_i} - \Delta p_n\, W_{t_n}, \qquad I_{(-1,1)} = \Delta p_n\, \Delta W_n - I_{(1,-1)}$$

- simulation jump times $\tau_i$ : $\quad W_{\tau_i} \Longrightarrow \quad I_{(1,-1)}$ and $I_{(-1,1)}$
- Computational effort heavily dependent on intensity $\lambda$

## Derivative-Free Strong Schemes

avoid computation of derivatives

$\Downarrow$

*order* **1.0** *derivative-free strong scheme*

## Implicit Strong Schemes

wide stability regions

$$\Downarrow$$

*implicit Euler scheme*

*order* $\mathbf{1.0}$ *implicit strong Taylor scheme*

# Predictor-Corrector Euler Scheme

- **corrector**

$$\begin{aligned} Y_{n+1} \;&=\; Y_n + \Big(\theta\, \bar{a}_\eta(\bar{Y}_{n+1}) + (1-\theta)\, \bar{a}_\eta(Y_n)\Big)\Delta_n \\ &\quad + \Big(\eta\, b(\bar{Y}_{n+1}) + (1-\eta)\, b(Y_n)\Big)\, \Delta W_n + \sum_{i=p(t_n)+1}^{p(t_{n+1})} c\,(\xi_i) \end{aligned}$$

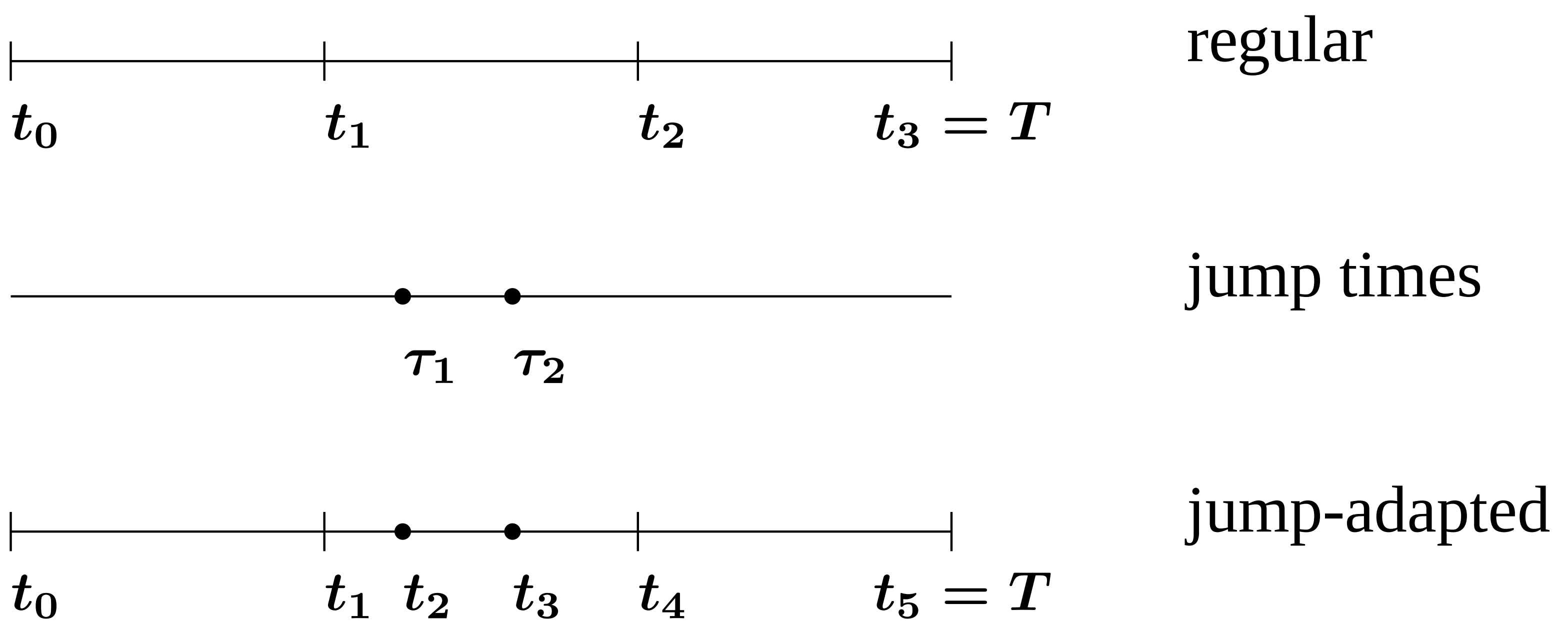
$$\bar{a}_\eta = a - \eta\, b\, b'$$

- **predictor**

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta_n + b(Y_n)\,\Delta W_n + \sum_{i=p(t_n)+1}^{p(t_{n+1})} c\,(\xi_i)$$

$\theta, \eta \in [0, 1]$ degree of implicitness

# Jump-Adapted Time Discretization



# Jump-Adapted Strong Approximations

jump-adapted time discretisation

$$\Downarrow$$

jump times included in time discretisation

- **jump-adapted Euler scheme**

$$Y_{t_{n+1}-} = Y_{t_n} + a(Y_{t_n})\Delta_{t_n} + b(Y_{t_n})\Delta W_{t_n}$$

and

$$Y_{t_{n+1}} = Y_{t_{n+1}-} + c(Y_{t_{n+1}-})\,\Delta p_n$$

- $\gamma = 0.5$

Merton SDE : $\mu = 0.05$, $\sigma = 0.2$, $\psi = -0.2$, $\lambda = 10$, $X_0 = 1$, $T = 1$

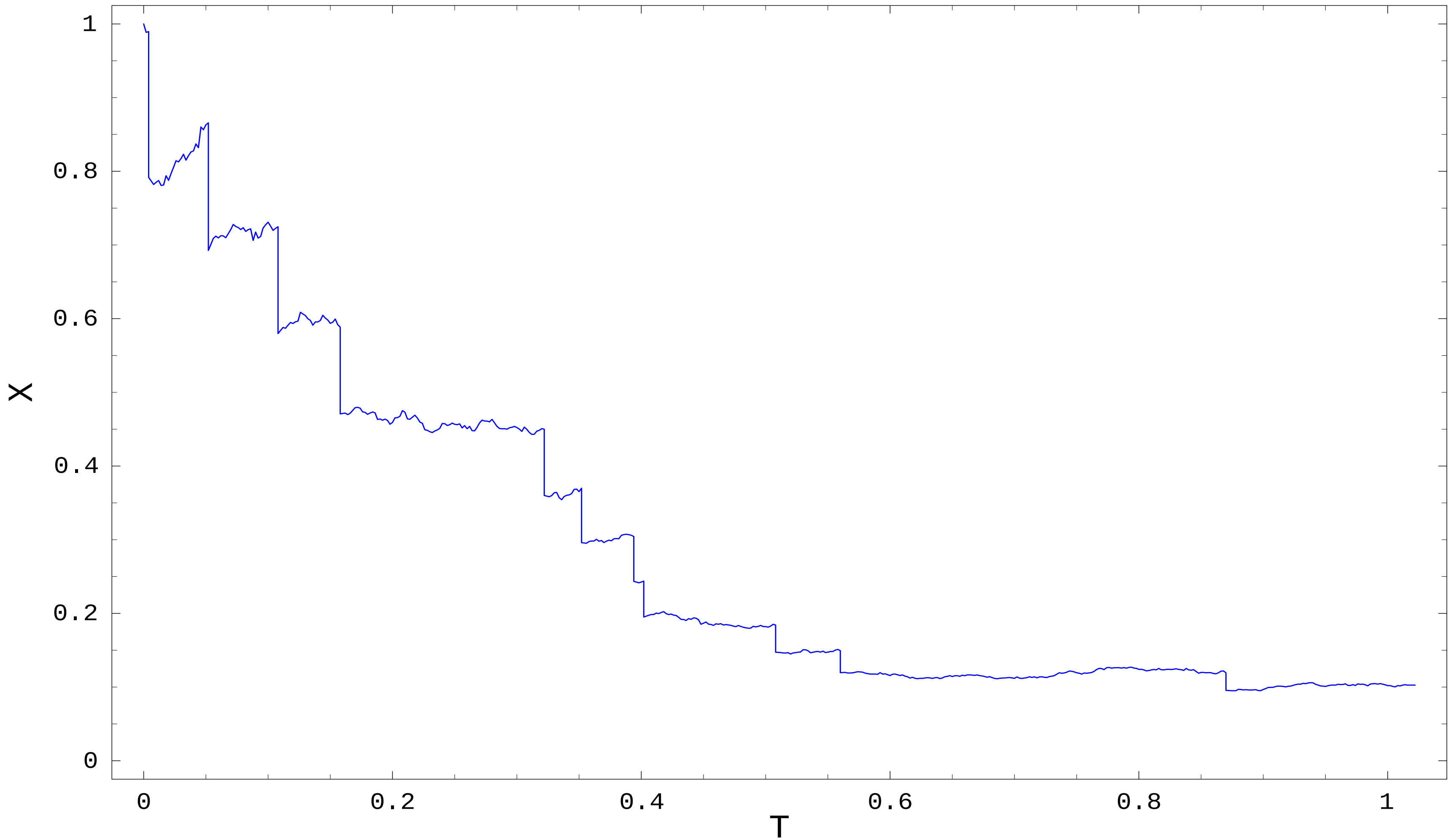



Figure 9: Plot of a jump-diffusion path.

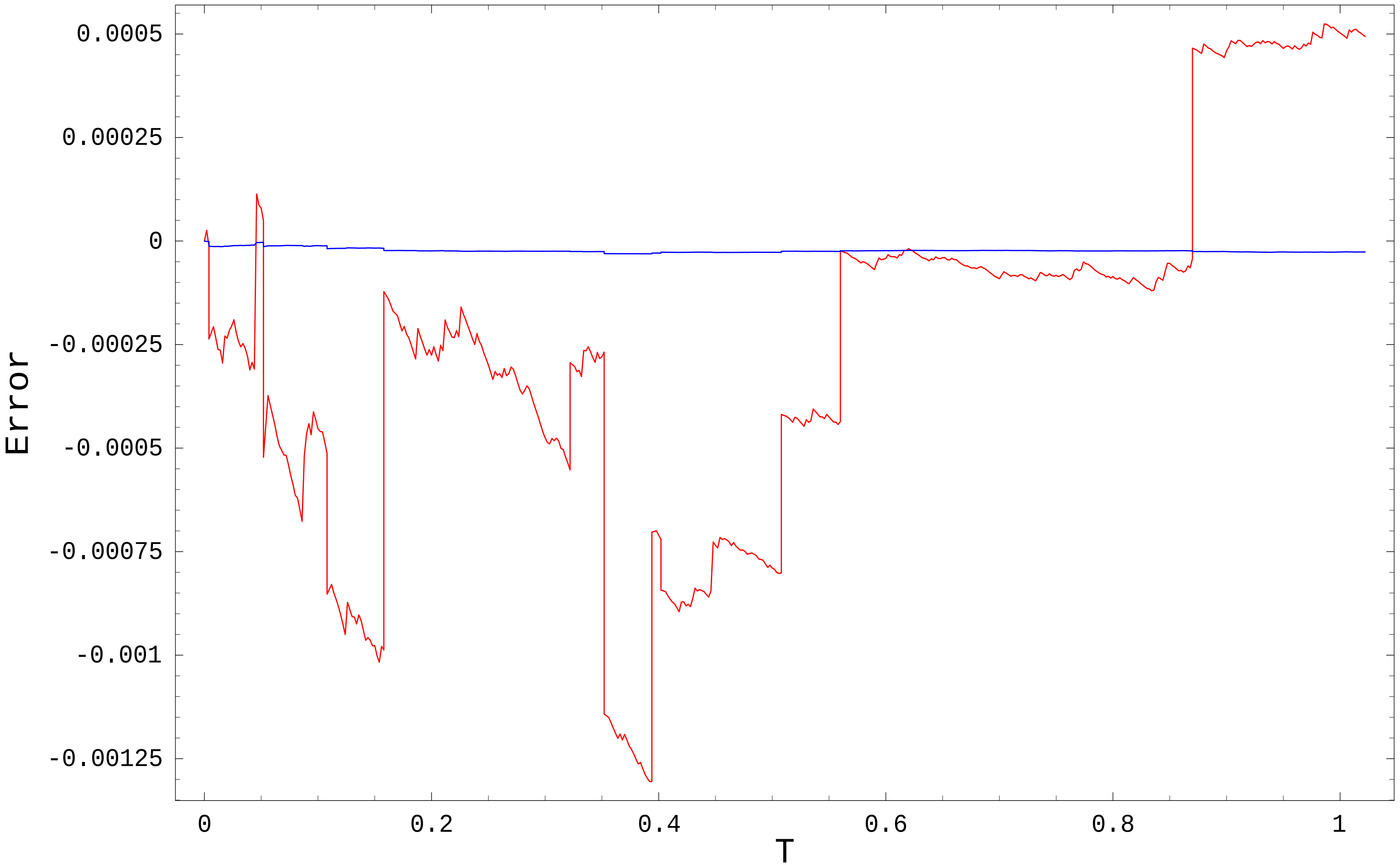



Figure 10: Plot of the strong error for Euler(red) and **1.0** Taylor(blue) scheme.

Merton SDE : $\boldsymbol{\mu = -0.05, \sigma = 0.1, \lambda = 1, X_0 = 1, T = 0.5}$

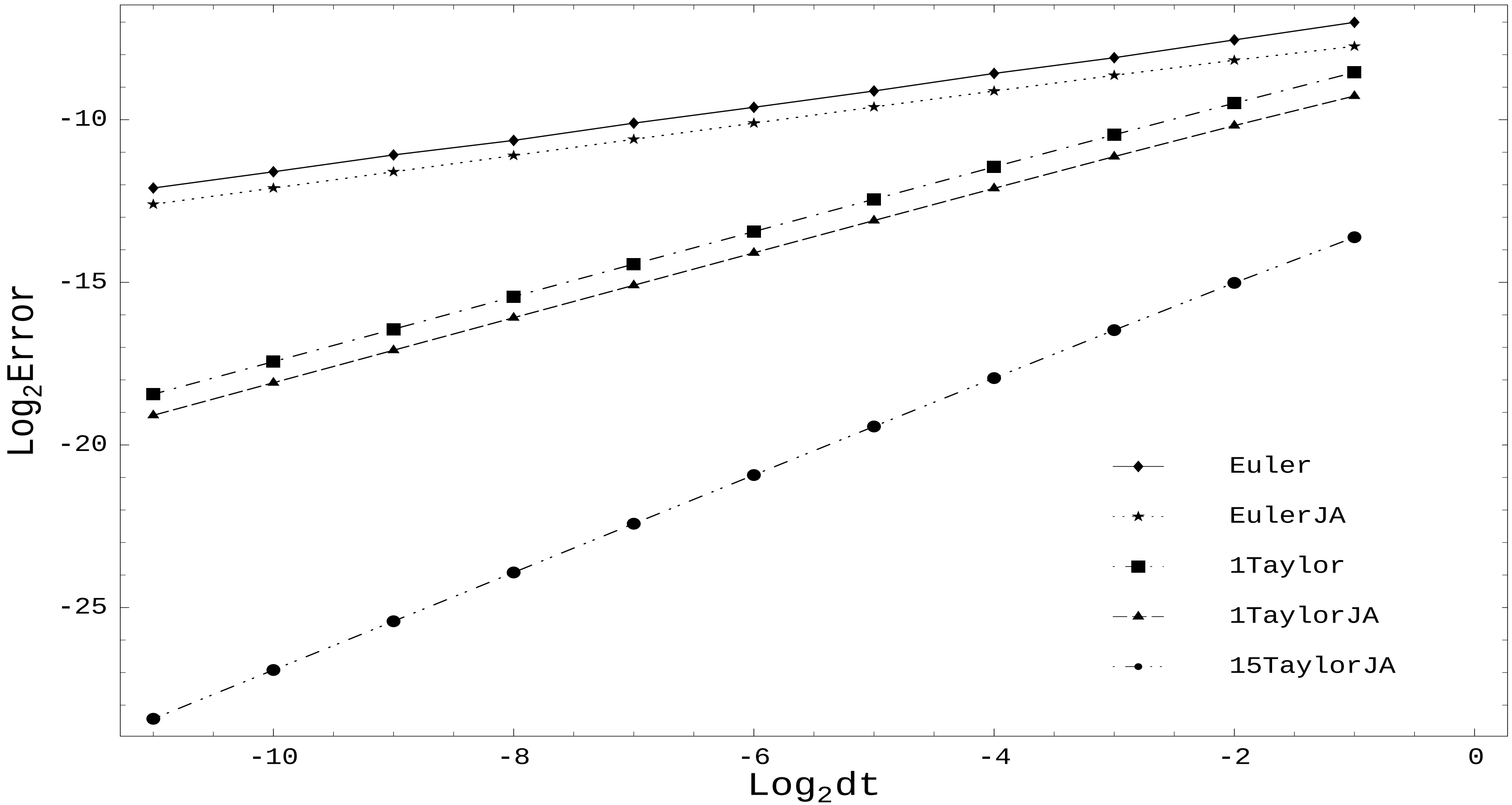



Figure 11: Log-log plot of strong error versus time step size.

## Literature on Weak Schemes with Jumps

- Mikulevicius & Pl (1991)
  $\Longrightarrow$ jump-adapted order $\boldsymbol{\beta} \in \{\mathbf{1}, \mathbf{2} \ldots\}$ weak schemes

- Liu & Li (2000) $\Longrightarrow$ order $\boldsymbol{\beta} \in \{\mathbf{1}, \mathbf{2} \ldots\}$ weak Taylor, extrapolation and simplified schemes

- Kubilius & Pl (2002) and Glasserman & Merener (2003)
  $\Longrightarrow$ jump-adapted Euler with weaker assumptions on coefficients

- Bruti-Liberati & Pl (2006) $\Longrightarrow$ jump-adapted order $\boldsymbol{\beta} \in \{\mathbf{1}, \mathbf{2} \ldots\}$ derivative-free, implicit and predictor-corrector schemes

## Simplified Euler Scheme

- **Euler scheme** $\Longrightarrow \quad \beta = 1$

- **simplified Euler scheme**

$$Y_{n+1} = Y_n + a(Y_n)\Delta + b(Y_n)\Delta\hat{W}_n + c(Y_n)\,(\hat{\xi}_n - 1)\Delta\hat{p}_n$$

- if $\Delta\hat{W}_n$ and $\Delta\hat{p}_n$ match the first 3 moments of $\Delta W_n$ and $\Delta p_n$ up to an $O(\Delta^2)$ error $\Longrightarrow \quad \beta = 1$

- $$P(\Delta\tilde{W}_n = \pm\sqrt{\Delta}) = \frac{1}{2}$$

# Jump-Adapted Taylor Approximations

- **jump-adapted Euler scheme** $\Longrightarrow \quad \beta = 1$

- **jump-adapted order 2 weak Taylor scheme**

$$
\begin{aligned}
Y_{t_{n+1}-} \; = \; & Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n} + \frac{b\,b'}{2}\Big( (\Delta W_{t_n})^2 - \Delta_{t_n}\Big) + a'\,b\,\Delta Z_{t_n} \\
& + \frac{1}{2}\left(a\,a' + \frac{1}{2}a''b^2\right)\Delta_{t_n}^2 + \left(a\,b' + \frac{1}{2}b''\,b^2\right)\{\Delta W_{t_n}\,\Delta_{t_n} - \Delta Z_{t_n}\}
\end{aligned}
$$

and

$$
Y_{t_{n+1}} = Y_{t_{n+1}-} + c(Y_{t_{n+1}-})\,\Delta p_n
$$

- $\beta = 2$

# Predictor-Corrector Schemes

- predictor-corrector $\Longrightarrow$ stability and efficiency

- **jump-adapted predictor-corrector Euler scheme**

$$Y_{t_{n+1}-} = Y_{t_n} + \frac{1}{2}\Big\{a(\bar{Y}_{t_{n+1}-}) + a\Big\}\Delta_{t_n} + b\Delta W_{t_n}$$

with predictor

$$\bar{Y}_{t_{n+1}-} = Y_{t_n} + a\Delta_{t_n} + b\Delta W_{t_n}$$

- $\beta = 1$

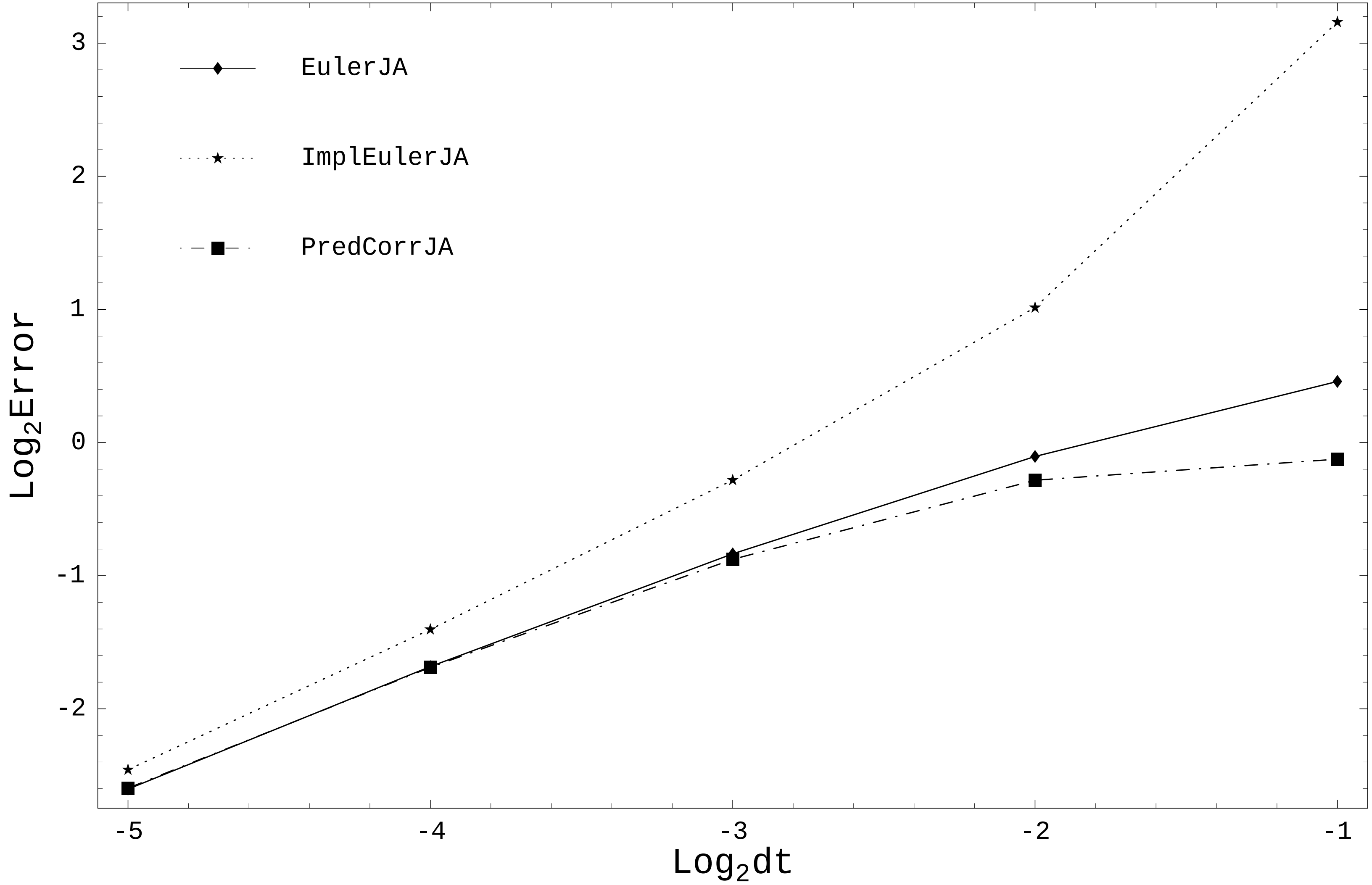



Figure 12: Log-log plot of weak error versus time step size.

## Regular Approximations

- higher order schemes : time, Wiener and Poisson multiple integrals

- random jump size difficult to handle

- higher order schemes: computational effort dependent on intensity

## Conclusions

- low intensity $\Longrightarrow$ jump-adapted higher order predictor-corrector
- high intensity $\Longrightarrow$ regular schemes
- distinction between strong and weak predictor-corrector schemes

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