Book Review: *Energy Derivatives: Pricing & Risk Management*, Clewlow L., Strickland C., 2000

Chapter 7, Spot Price Models: Pricing Path Dependent and American Style Options

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Outline

- Monte Carlo Simulations
  - Single Factor Models
  - Multi-Factor Models
- Trinomial Trees
  - Implied Trinomial Trees
  - Pricing General Path Dependent Options
  - Pricing Asian Options
  - Pricing Swing Options

Single Factor Model Simulations

- Monte Carlo Simulation

$$\hat{C}_t = \frac{1}{M}\sum_{j=1}^{M} C_{t,j}$$

- Single Factor Simulation (Schwartz model)
  - Logprice: $dx = \alpha(\hat{\mu} - x)dt + \sigma dz$
  - Discretised: $\Delta x = \alpha(\hat{\mu} - x)\Delta t + \sigma \Delta z$
  - Δz : Simulated by $\sqrt{\Delta t}\varepsilon$
  - Simulated price: $S_i = \exp(x_i)$

$$x_i = x_{i-1} + \alpha(\hat{\mu} - x_{i-1})\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i$$

Single Factor Model Simulations

- Futures Option Pricing (Schwartz model)
  - Payoff: $C_T = \max(0, F(T,s) - K)$
    - $F(T,s)$ can be calculated analytically
  - Simulation:

$$\hat{C}_t = P(t,T)\frac{1}{M}\sum_{j=1}^{M}\max(0, F_j(T,s) - K)$$

Single Factor Model Simulations

- European Futures Options: Results

S	α	μ	λ	σ	μhat	K	T	s	N
26.90	0.472	2.925	0.000	0.368	2.782	23.20	0.50	1.00	10

dt	ln(S)	P(0,T)	call_value	SE	M
0.05	3.292	0.951	1.616	0.091	1000

t	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
ε		0.708	0.574	-0.203	-0.006	-0.027	-0.013	0.367	0.629	0.434	-1.140
xt	3.292	3.338	3.373	3.342	3.328	3.313	3.300	3.318	3.357	3.379	3.271

ST	F(T,s)	CT
26.34	24.41	1.22

FIGURE 7.1 **Monte Carlo Simulation of Schwartz (1997) 1 Factor Model for a European Futures Option**

Single Factor Model Simulations

- Variance Reduction Techniques
  - Antithetic variants

$$\bar{S}_i = \exp(\bar{x}_i)$$

$$\bar{x}_i = \bar{x}_{i-1} + \alpha(\hat{\mu} - \bar{x}_{i-1})\Delta t + \sigma\sqrt{\Delta t}(-\varepsilon_i)$$

$$C_{T,j} = \max\{0, F(T,s,S_T) - K\}$$

$$\bar{C}_{T,j} = \max\{0, F(T,s,\bar{S}_T) - K\}$$

S	α	μ	λ	σ	μhat	K	T	s	N
26.90	0.472	2.925	0.000	0.368	2.782	23.20	0.50	1.00	10

dt	ln(S)	P(0,T)	call_value	SE	M
0.05	3.292	0.951	1.628	0.052	1000

t	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
ε		0.708	0.574	-0.203	-0.006	-0.027	-0.013	0.367	0.629	0.434	-1.140
xt	3.292	3.338	3.373	3.342	3.328	3.313	3.300	3.318	3.357	3.379	3.271
xtbar	3.292	3.222	3.164	3.172	3.163	3.156	3.149	3.110	3.050	3.008	3.097

ST	F(T,s)	CT
26.34	24.41	1.22
22.12	21.27	0.00

FIGURE 7.3 **Monte Carlo Simulation (with Antithetics) of Schwartz (1997) 1 Factor Model for a European Futures Option**

Single Factor Model Simulations

- Convergence of Monte Carlo Simulations

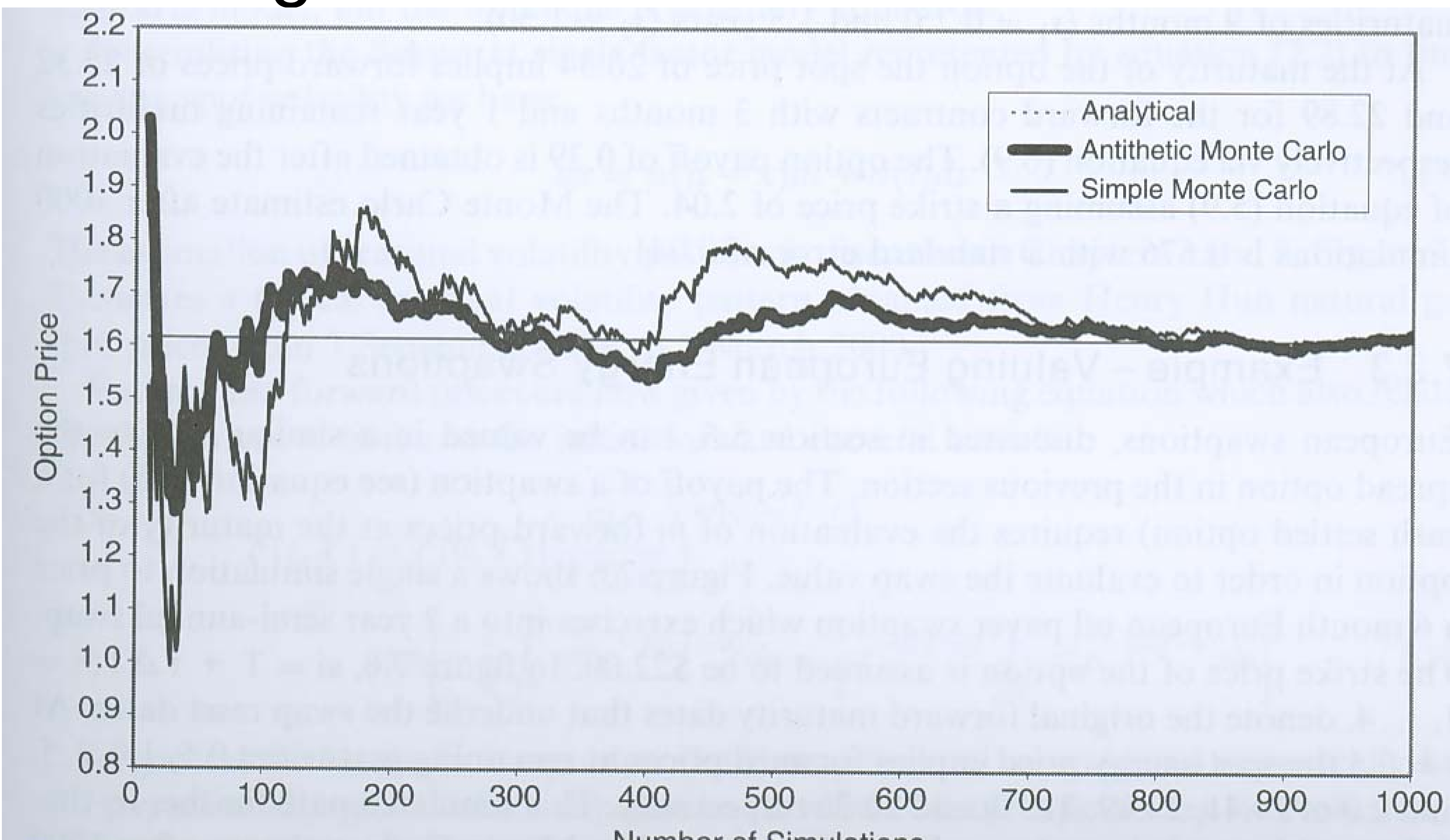

FIGURE 7.4 Convergence of Monte Carlo Simulation With Antithetics for European Futures Option

Single Factor Model Simulations

- Valuing Spread Options
  $max(0, F(T,s_1)-F(T,s_2)-K)$

S	α	μ	λ	σ	μhat	K	T	s1	s2	N
26.90	0.472	2.925	0.000	0.368	2.782	2.04	0.50	0.75	1.50	10

dt	ln(S)	P(0,T)	call_value	SE	M
0.05	3.292	0.951	0.676	0.041	1000

t	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
ε		0.708	0.574	-0.203	-0.006	-0.027	-0.013	0.367	0.629	0.434	-1.140
xt	3.292	3.338	3.373	3.342	3.328	3.313	3.300	3.318	3.357	3.379	3.271

ST	F(T,s1)	F(T,s2)	CT
26.34	25.32	22.89	0.39

FIGURE 7.5 Monte Carlo Simulation of Schwartz (1997) 1 Factor Model for a European Calendar Spread Option

Single Factor Model Simulations

- Valuing European Energy Swaptions

$$\max(0, \frac{1}{m}\sum_{k=1}^{m}(F(T, s_k) - K))$$

S	α	μ	λ	σ	μhat	K	T	ΔT	N
26.90	0.472	2.925	0.000	0.368	2.782	22.00	0.50	0.50	10

dt	ln(S)	P(0,T)	call_value	SE	M
0.05	3.292	0.951	0.922	0.056	1000

t	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
ε		0.708	0.574	-0.203	-0.006	-0.027	-0.013	0.367	0.629	0.434	-1.140
xt	3.292	3.338	3.373	3.342	3.328	3.313	3.300	3.318	3.357	3.379	3.271

ST	F(T,s1)	F(T,s2)	F(T,s3)	F(T,s4)	CT
26.34	24.41	22.89	21.70	20.76	0.44

FIGURE 7.6 Monte Carlo Simulation of Schwartz (1997) 1 Factor Model for a European Energy Swaption

Single Factor Model Simulations

- Incorporating Seasonality into Simulations

$$dx = \alpha(\hat{\mu} - x)dt + \sigma(t)dz$$

$$F(t,T) = F(0,T)\left(\frac{S(t)}{F(0,t)}\right)^{\exp[-\alpha(T-t)]}$$

$$\exp\left[-\frac{1}{2}\int_0^t \sigma_u^2 e^{-2\alpha(T-u)}du + \frac{1}{2}e^{-\alpha(T-t)}\int_0^t \sigma_u^2 e^{-2\alpha(t-u)}du\right]$$

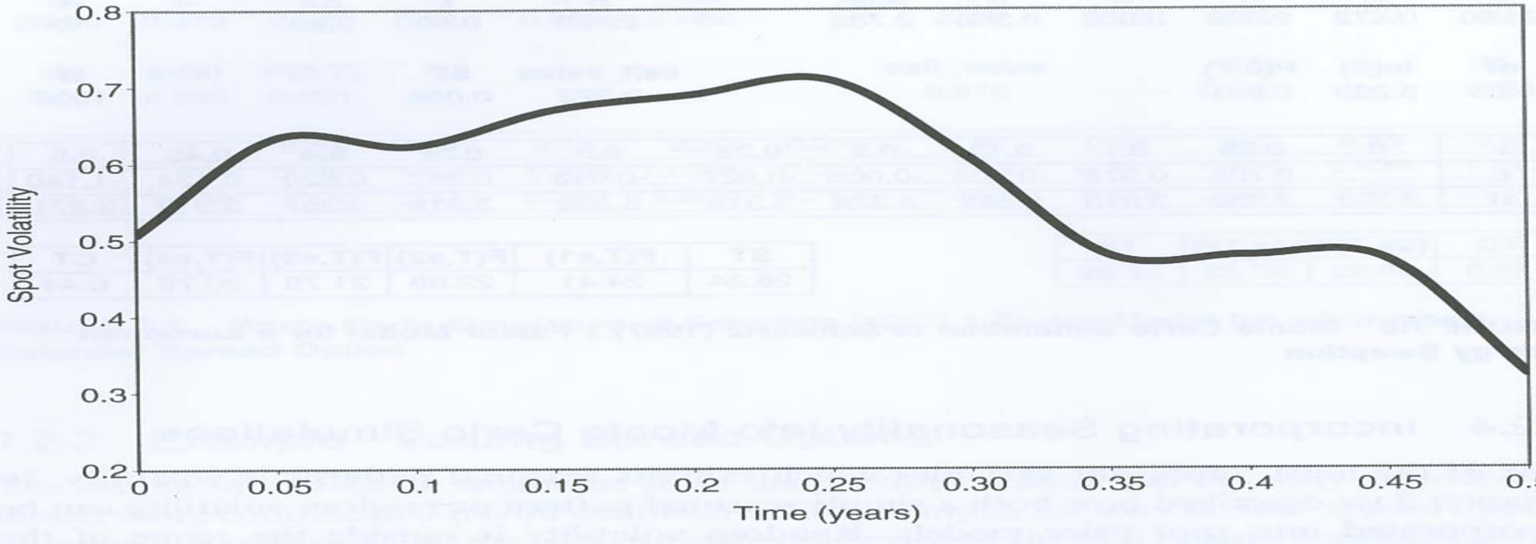

FIGURE 7.7 Typical Seasonal Energy Spot Price Volatility Pattern for Henry Hub Natural Gas

Single Factor Model Simulations

- Valuing European Futures Option with Seasonal Volatility

S	α	μ	λ	μhat	K	T	s	N
26.90	0.472	2.925	0.000	2.782	23.20	0.50	1.00	10

dt	ln(S)	P(0,T)	call_value	SE	M
0.05	3.292	0.951	2.955	0.169	1000

t	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
$\sigma(t)$	0.508	0.633	0.623	0.669	0.689	0.708	0.606	0.479	0.478	0.472	0.322
ε		0.708	0.574	-0.203	-0.006	-0.027	-0.013	0.367	0.629	0.434	-1.140
xt	3.292	3.361	3.428	3.385	3.370	3.352	3.336	3.373	3.426	3.457	3.321

ST	F(T,s)	CT
27.69	25.40	2.20

FIGURE 7.8 Monte Carlo Simulation of Schwartz (1997) 1 Factor Model for a European Futures Option with Seasonal Volatility

Single Factor Model Simulations

- Computation of Hedge Sensitivities
  - Finite difference approximation

$$delta = \frac{\partial C}{\partial S} \approx \frac{C(S + \Delta S) - C(S - \Delta S)}{2\Delta S}$$

$$gamma = \frac{\partial^2 C}{\partial S^2} \approx \frac{C(S + \Delta S) - 2C(S) + C(S - \Delta S)}{\Delta S^2}$$

$$vega = \frac{\partial C}{\partial \sigma} \approx \frac{C(\sigma + \Delta \sigma) - C(\sigma - \Delta \sigma)}{2\Delta \sigma}$$

$$theta = \frac{\partial C}{\partial t} \approx \frac{C(t + \Delta t) - C(t)}{\Delta t}$$

$$rho = \frac{\partial C}{\partial r} \approx \frac{C(r + \Delta r) - C(r - \Delta r)}{2\Delta r}$$

Multi-Factor Model Simulations

- Simulation of Multi-Factor Models (Schwartz model)
  - $$dS = (r - \delta)S dt + \sigma S dz$$
    $$d\delta = \alpha_\delta(\bar{\delta} - \delta)dt + \sigma_\delta dz_\delta$$
    dz and dz_δ have instantaneous correlation $\rho_{S\delta}$.
  - $$\Delta S = (r - \delta)S\Delta t + \sigma S \Delta z$$
    $$\Delta\delta = \alpha_\delta(\bar{\delta} - \delta)\Delta t + \sigma_\delta \Delta z_\delta$$
    $$\Delta z = \varepsilon_1 \sqrt{\Delta t}$$
    $$\Delta z_\delta = \left(\rho_{S\delta}\varepsilon_1 + \sqrt{1 - \rho_{S\delta}^2}\varepsilon_2\right)\sqrt{\Delta t}$$

Multi-Factor Model Simulations

- Logprices

$$x_{i+1} = x_i + (r - \delta_i - \tfrac{1}{2}\sigma^2)\Delta t + \sigma \varepsilon_1 \sqrt{\Delta t}$$

$$\delta_{i+1} = \delta_i + \alpha_\delta(\bar{\delta} - \delta_i)\Delta t + \sigma_\delta\left(\rho_{S\delta}\varepsilon_1 + \sqrt{1 - \rho_{S\delta}^2}\varepsilon_2\right)\sqrt{\Delta t}$$

- Results

S	r	δ	σ	αδ	δbar	σδ	ρSδ	K	T	s
26.90	0.10	0.266	0.368	0.363	0.171	0.301	0.221	23.15	0.5	1.00

dt	ln(S)	call_value	SE	N	M
0.05	3.292	2.251	0.139	10	1000

t	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
ε1		0.691	-0.100	0.042	0.211	-0.535	-0.156	0.940	0.301	-0.522	2.464
ε2		0.879	2.244	-0.233	-0.727	0.081	-1.554	-0.860	-0.788	-0.714	-1.439
xt	3.292	3.337	3.314	3.295	3.292	3.229	3.198	3.262	3.276	3.225	3.423
δ	0.266	0.332	0.475	0.455	0.405	0.398	0.290	0.245	0.197	0.142	0.084

ST	F(T,s)	CT
30.65	30.74	7.59

Simulation Methods

- Generation of the Random Numbers
  - Polar rejection method

repeat

$x_1 = 2 \times \text{standard\_ uniform\_ random\_ number} - 1$

$x_2 = 2 \times \text{standard\_ uniform\_ random\_ number} - 1$

$w = x_1^2 + x_2^2$

until $w > 1$

$$c = \sqrt{-2\frac{\ln(w)}{w}}$$

$z_1 = cx_1$

$z_2 = cx_2$

  - Quasi Monte Carlo

Simulation Methods

- Valuing Path Dependent Options
  - Barrier options

$$\max(0, F(T,s) - K)1_{\min(F(t_1,s),\ldots,F(t_m,s))>H}$$

S	α	μ	λ	σ	μhat	K	T	s	H	N
26.90	0.472	2.925	0.000	0.368	2.782	23.20	0.50	1.00	22.04	10

dt	ln(S)	P(0,T)	F(0,s)	call_value	SE	M
0.05	3.292	0.951	23.20	1.264	0.083	1000

t	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
ε		0.417	-0.877	2.114	-0.969	-0.920	1.609	-0.644	-2.190	1.049	0.799
xt	3.292	3.314	3.230	3.393	3.299	3.211	3.333	3.267	3.075	3.155	3.212
F(t,s)	23.20	23.68	22.55	25.30	23.91	22.64	24.85	23.86	**20.78**	22.17	23.30

CT
0.00

FIGURE 7.10 Monte Carlo Simulation of Schwartz (1997) 1 Factor Model for a European Barrier Option

Simulation Methods

- Valuing Path Dependent Options
  - Lookback options

$$\max(0, F(T,s) - \min(F(t_1,s),....,F(t_m,s)))$$

S	α	μ	λ	σ	μhat	T	s	N
26.90	0.472	2.925	0.000	0.368	2.782	0.50	1.00	10

dt	ln(S)	P(0,T)	F(0,s)	call_value	SE	M
0.05	3.292	0.951	23.20	2.495	0.089	1000

t	**0**	**0.05**	**0.1**	**0.15**	**0.2**	**0.25**	**0.3**	**0.35**	**0.4**	**0.45**	**0.5**
ε		-0.200	-0.519	0.503	1.427	-1.359	0.164	-1.619	-0.993	0.266	0.246
xt	3.292	3.264	3.210	3.241	3.348	3.222	3.225	3.082	2.993	3.010	3.025
F(t,s)	23.20	22.92	22.26	22.85	24.72	22.82	23.00	20.81	**19.53**	19.82	20.10

CT
0.57

FIGURE 7.11 Monte Carlo Simulation of Schwartz (1997) 1 Factor Model for a European Lookback Option

Simulation Methods

- Valuing Path Dependent Options
  - Asian options

$$\max(0, \frac{1}{m}\sum_{k=1}^{m} F(t_k, s) - K)$$

S	α	μ	λ	σ	μhat	K	T	s	N
26.90	0.472	2.925	0.000	0.368	2.782	23.20	0.50	1.00	10

dt	ln(S)	P(0,T)	F(0,s)	call_value	SE	M
0.05	3.292	0.951	23.20	0.891	0.048	1000

t	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
ε		0.708	0.574	-0.203	-0.006	-0.027	-0.013	0.367	0.629	0.434	-1.140
xt	3.292	3.338	3.373	3.342	3.328	3.313	3.300	3.318	3.357	3.379	3.271
F(t,s)	23.199	24.045	24.762	24.448	24.398	24.319	24.256	24.755	25.688	26.351	24.414

CT
1.40

FIGURE 7.12 Monte Carlo Simulation of Schwartz (1997) 1 Factor Model for a European Asian Call Option

Trinomial Trees

- Revised Single Factor Schwartz Model
  - Consistent with observed forward curve

$$dx(t) = [\theta(t) - \alpha x(t)]dt + \sigma dz(t)$$

$$\theta(t) = \left(\frac{\partial \ln F(0,t)}{\partial t} + \alpha \ln F(0,t) + \frac{\sigma^2}{4}\left(1 - e^{-2\alpha t}\right) - \frac{1}{2}\sigma^2 \right)$$

  - Building trees with $\theta(t) = 0$

$$d\bar{x}(t) = -\alpha \bar{x}(t)dt + \sigma dz(t)$$

    - Space step: $\Delta \bar{x} = \sigma\sqrt{3\Delta t}$

Trinomial Trees

- Transition probabilities

Let $p_{u,i,j}$, $p_{m,i,j}$, and $p_{d,i,j}$ define the probabilities associated with the lower, middle, and upper branches emanating from node (i, j) respectively. These probabilities are given by:

$$p_{u,i,j} = \frac{1}{2}\left[\frac{\sigma^2\Delta t + \alpha^2 x_{i,j}^2 \Delta t^2}{\Delta x^2} + (k-j)^2 - \frac{\alpha x_{i,j}\Delta t}{\Delta x}(1 + 2(k-j)) + (k-j)\right]$$

$$p_{d,i,j} = \frac{1}{2}\left[\frac{\sigma^2\Delta t + \alpha^2 x_{i,j}^2 \Delta t^2}{\Delta x^2} + (k-j)^2 + \frac{\alpha x_{i,j}\Delta t}{\Delta x}(1 + 2(k-j)) + (k-j)\right] \qquad (7.30)$$

$$p_{m,i,j} = 1 - p_{u,i,j} - p_{d,i,j}$$

- Make Trees Consistent with Future Curves
  – Displacing nodes at $i\Delta t$ by a_i

Trinomial Trees

- Achieving a_i
  - Define $Q_{i,j}$ as the value, at time 0, of a security that pays $1 if node *(i,j)* is reached, or $0 otherwise. (Arrow-Debreu securities)

$$Q_{i+1,j} = \sum_{j'} Q_{i,j'} p_{j',j} P(i\Delta t, (i+1)\Delta t)$$

where $p_{j',j}$ is the probability of moving from node (i, j') to node $(i+1, j)$ and $P(i\Delta t, (i+1)\Delta t)$ denotes the price at time $i\Delta t$ of the pure discount bond maturing at time $(i+1)\Delta t$.

  - $$a_i = \ln\left(\frac{P(0, i\Delta t)F(0, i\Delta t)}{\sum_j Q_{i,j} e^{\bar{x}_{i,j}}}\right)$$

Market Forward Curves

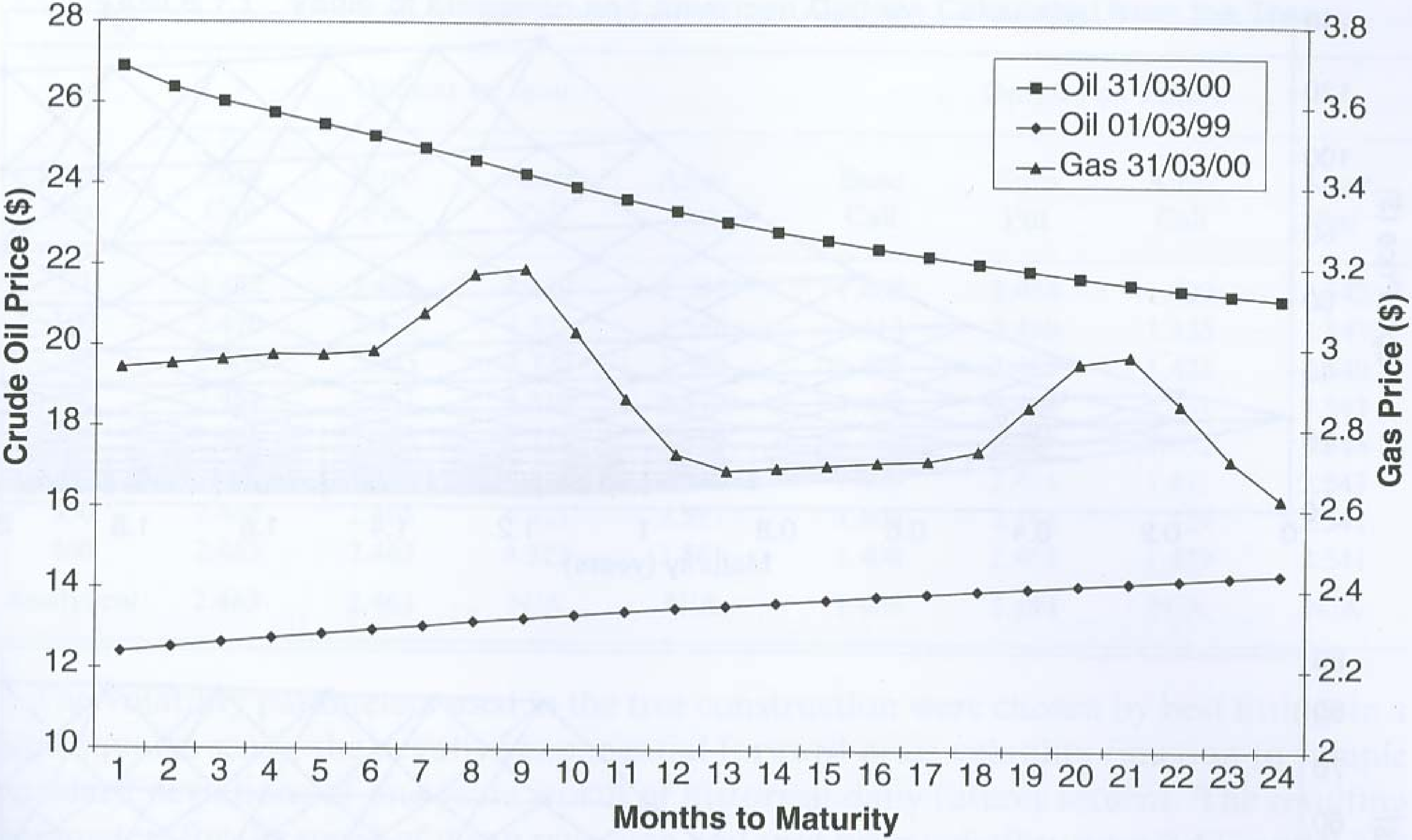

FIGURE 7.13 Typical Energy Market Forward Curves

Fitted Spot Price Trees

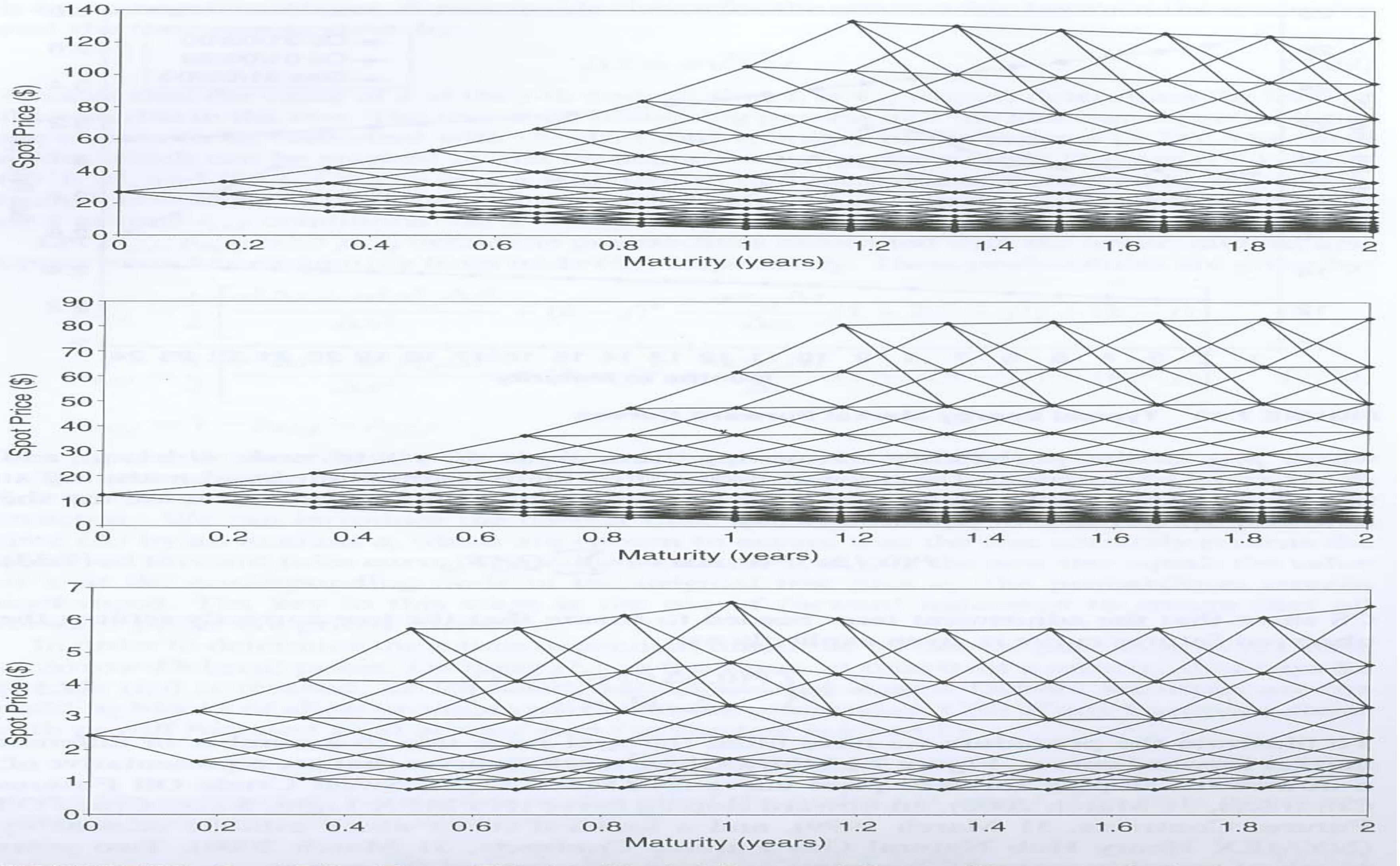

FIGURE 7.14 Spot Price Trees Fitted to Market Forward Curves (Downward sloping, Upward Sloping, and Seasonal)

Trinomial Trees

TABLE 7.1 Value of European and American Options Calculated from the Tree

	Options on Spot				Options on Future			
Steps/ Year	Euro Call	Euro Put	Amer Call	Amer Put	Euro Call	Euro Put	Amer Call	Amer Put
50	2.482	2.482	4.320	2.596	1.408	2.484	1.429	2.542
100	2.470	2.470	4.322	2.586	1.413	2.490	1.435	2.547
150	2.465	2.465	4.320	2.580	1.406	2.483	1.428	2.540
200	2.461	2.461	4.319	2.577	1.409	2.486	1.431	2.543
250	2.463	2.463	4.321	2.579	1.410	2.487	1.432	2.544
300	2.464	2.464	4.322	2.580	1.409	2.485	1.431	2.543
350	2.465	2.465	4.321	2.581	1.406	2.483	1.429	2.541
400	2.463	2.463	4.321	2.581	1.408	2.484	1.429	2.541
Analytical	2.463	2.463	N/A	N/A	1.408	2.484	N/A	N/A

Implied Trinomial Tress

- Generalized Trinomial Trees
- Making Constant Parameters (e.g. probabilities) Time Dependent
- See Chapter 5, *Implementing Derivatives Models*, by Clewlow and Strickland, 1998, John Wiley, London

Pricing General Path Dependent Energy Options in Spot Price Trees

- General Base Function:

$$G(F(t,s); 0 \leq t \leq T, t \leq s)$$

- Values at Maturity:

$$C_{N,j,k} = C(t_N, G_{N,j,k});\ \forall j,k$$

- Backward Programming:

$$C_{i,j,k} = e^{-f(i,i+1)\Delta t}\left(p_{u,i,j}C_{i+1,j+1,u} + p_{m,i,j}C_{i+1,j,m} + p_{d,i,j}C_{i+1,j-1,d}\right)$$

Pricing Asian Energy Options in Spot Price Trees

- Pricing Asian Energy Options
  - Average taken on fixed dates $t_l, l = 1, \ldots, L.$
  - Base function:

$$G_{i,j,1} = \begin{cases} \frac{G_{i-1,j_l,1} m_{i-1} + S_j}{m_i} & \text{if} \quad t_i = t_{m_i} \text{ i.e. a fixing date} \\ G_{i-1,j_l,1} & \text{otherwise} \end{cases}$$

$$G_{i,j,n} = \begin{cases} \frac{G_{i-1,j_u,n} m_{i-1} + S_j}{m_i} & \text{if} \quad t_i = t_{m_i} \text{ i.e. a fixing date} \\ G_{i-1,j_u,n} & \text{otherwise} \end{cases}$$

$$G_{i,j,k} = G_{i,j,1} e^{(k-1)h}$$

$$h = \frac{\ln(G_{i,j,n}) - \ln(G_{i,j,1})}{n_{i,j} - 1}$$

Pricing Asian Energy Options in Spot Price Trees

- Branches

$$G_{i+1,j+1,u} = \begin{cases} \frac{G_{i,j,k} m_i + S_{j+1}}{m_{i+1}} & \text{if } \; t_{i+1} = t_{m_{i+1}} \text{ i.e. a fixing date} \\ G_{i,j,k} & \text{otherwise} \end{cases}$$

$$G_{i+1,j,m} = \begin{cases} \frac{G_{i,j,k} m_i + S_j}{m_{i+1}} & \text{if } \; t_{i+1} = t_{m_{i+1}} \text{ i.e. a fixing date} \\ G_{i,j,k} & \text{otherwise} \end{cases}$$

$$G_{i+1,j-1,d} = \begin{cases} \frac{G_{i,j,k} m_i + S_{j-1}}{m_{i+1}} & \text{if } \; t_{i+1} = t_{m_{i+1}} \text{ i.e. a fixing date} \\ G_{i,j,k} & \text{otherwise} \end{cases}$$

Pricing Asian Energy Options in Spot Price Trees

TABLE 7.2 Convergence of European and American Fixed Strike Average Price Call Options

	Number of Averages at Each Node of Tree ($n_{i,j}$)					
	4		12		20	
Steps/Year	Euro	Amer	Euro	Amer	Euro	Amer
50	2.526	2.702	2.488	2.671	2.484	2.667
100	2.575	2.789	2.491	2.711	2.484	2.704
150	2.629	2.873	2.499	2.737	2.486	2.726
200	2.684	2.953	2.507	2.755	2.488	2.738
250	2.738	3.033	2.520	2.773	2.490	2.746
300	2.792	3.110	2.534	2.793	2.495	2.753
350	2.874	3.225	2.559	2.830	2.505	2.771
400	2.926	3.297	2.575	2.852	2.513	2.780

Pricing Swing Options in Spot Price Trees

- "Take-or-Pay" Option
  - Dates: $t_i;\ i = 1, \ldots, m$
  - Strike prices: K_i
  - Maximum volume each day: $V_{\max},$
  - Minimum overall volume: $V_{\min}$
    - Penalty price for volume less than $V_{\min}$: $(1 - z)K_N$
- Decision at Maturity

$$V^*_{N,j,k} = \begin{cases} V_{\max} & \text{for} \quad S_{N,j,k} > K_N \\ \min(V_{N,j,k}, V_{\max}) & \text{for} \quad K_N > S_{N,j,k} \geq (1 - z)K_N \\ 0 & \text{for} \quad (1 - z)K_N > S_{N,j,k} \geq 0 \end{cases}$$

$$C^*_{N,j,k} = (S_{N,j,k} - K_N)V^*_{N,j,k} - zK_N \max(0, V_{N,j,k} - V^*_{N,j,k})$$

Summary

- Monte Carlo Simulations:
  Flexible for path dependent options, can easily be extended to handle seasonality and to implement multi-factor models
- Trinomial Trees:
  Allows pricing options with early exercise opportunities