BSDEs in Discrete time and their extensions

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Outline

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BSDEs

A BSDE is an equation of the form

$$Y_t - \int_{]t,T]} F(\omega, u, Y_u, Z_u) du + \int_{]t,T]} Z_u dW_u = Q$$

where the solution pair (Y, Z) is adapted, Z is predictable and Q is some $\mathcal{F}_T$-measurable random variable.

- These equations have been studied in depth over the last 20 years.
- They have significant applications in Optimal Control and Mathematical Finance.
- My interest is on generalising these equations to allow for different types of filtrations and randomness.

- We shall initially consider discrete time processes satisfying ‘Backward Stochastic Difference Equations’.
- These are the natural extension of Backward Stochastic Differential Equations to discrete time.
- In this context, we shall show
  - Necessary and sufficient conditions for existence and uniqueness
  - A version of the comparison theorem
  - A representation of *every* dynamic nonlinear expectation as solution of a BSDE with certain properties, and conversely.

A probabilistic setting

- Let X be a discrete time, finite state process. Without loss of generality, X takes values from the unit vectors in $\mathbb{R}^N$.
- Let $\{\mathcal{F}_t\}$ be the filtration generated by X, that is $\mathcal{F}_t$ consists of every event that can be known from watching X up to time t.
- Let $M_t = X_t - E[X_t|\mathcal{F}_{t-1}]$. Then M is a martingale difference process, that is $E[M_t|\mathcal{F}_{t-1}] = \mathbf{0} \in \mathbb{R}^N$.

Discrete BSDEs ('D=Difference')

A BSDE is an equation of the form:

$$Y_t - \sum_{t \leq u < T} F(\omega, u, Y_u, Z_u) + \sum_{t \leq u < T} Z_u M_{u+1} = Q$$

- Q is the terminal condition (in $L^1(\mathcal{F}_T; \mathbb{R})$)
- F is a (stochastic) 'driver' function, with $F(\omega, u, \cdot, \cdot)$ known at time u.
- A solution is an adapted pair (Y, Z) of processes,

$$Y_t(\omega) \in \mathbb{R} \text{ and } Z_t(\omega) \in \mathbb{R}^{N \times 1}.$$

- All quantities are assumed to be $\mathbb{P}$-a.s. finite.
  (Note $L^0 = L^1 = L^2 = L^\infty$)

Discrete BSDEs ('D=Difference')

Equivalently, we can write this in a differenced form:

$$Y_t - F(\omega, t, Y_t, Z_t) + Z_t M_{t+1} = Y_{t+1}$$

with terminal condition

$$Y_T = Q.$$

The important detail is that

- The *terminal* condition is fixed, and the dynamics are given in reverse.
- The solution (Y, Z) is adapted, that is, at time t it depends only on what has happened up to time t.

A Special Case: Binomial Pricing

- Suppose we have a market with two assets: a stock Y following a simple binomial price process, and a risk free Bond B.
- Let r_t denote the one-step interest rate at time t.
- From each time t, there are two possible states for the stock price the following day, $Y(t+1,\uparrow)$ and $Y(t+1,\downarrow)$.
- Suppose these two states occur with (real world) probabilities p and $1-p$ respectively.

It is easy to show that there exists a unique ‘no-arbitrage’ price

$$\begin{aligned} Y(t) &= \frac{1}{1+r_t}\left[\pi Y(t+1,\uparrow) + (1-\pi) Y(t+1,\downarrow)\right] \\ &= \frac{1}{1+r_t} E_\pi[Y(t+1)|\mathcal{F}_t]. \end{aligned}$$

Here π the ‘risk-neutral probability’.

Writing $Y_t = Y(t)$ etc... we also know,

$$Y_{t+1} = E_p[Y_{t+1}|\mathcal{F}_t] + L_{t+1}$$

where $E_p[Y_{t+1}|\mathcal{F}_t]$ is the (real-world) conditional mean value of Y_{t+1}, and L_{t+1} is a random variable with conditional mean value zero

$$(L_{t+1} = Y_{t+1} - E_p[Y_{t+1}|\mathcal{F}_t]).$$

In the notation we established before, we can define a martingale difference process M

$$M_{t+1}(\uparrow) = \begin{bmatrix} 1-p \\ p-1 \end{bmatrix}, M_{t+1}(\downarrow) = \begin{bmatrix} -p \\ p \end{bmatrix}.$$

And it is easy to show that L_{t+1} can be written as $Z_t M_{t+1}$, for some row vector Z_t known at time t.

We can then do some basic algebra:

$$
\begin{aligned}
Y_{t+1} &= E_p[Y_{t+1}|\mathcal{F}_t] + L_{t+1} \\
&= Y_t + r_t Y_t - (1+r_t) Y_t + E_p[Y_{t+1}|\mathcal{F}_t] + Z_t M_{t+1} \\
&= Y_t + r_t Y_t - (1+r_t)\frac{1}{1+r_t} E_\pi[Y_{t+1}|\mathcal{F}_t] + E_p[Y_{t+1}|\mathcal{F}_t] + Z_t M_{t+1} \\
&= Y_t + r_t Y_t - E_\pi[Y_{t+1} - E_p(Y_{t+1})|\mathcal{F}_t] + Z_t M_{t+1} \\
&= Y_t + r_t Y_t - E_\pi[L_{t+1}|\mathcal{F}_t] + Z_t M_{t+1} \\
&= Y_t - \Big(- r_t Y_t + Z_t E_\pi[M_{t+1}|\mathcal{F}_t]\Big) + Z_t M_{t+1} \\
&= Y_t - F(Y_t, Z_t) + Z_t M_{t+1}
\end{aligned}
$$

So our one-step pricing formula is equivalent to the equation

$$Y_{t+1} = Y_t - F(Y_t, Z_t) + Z_t M_{t+1}$$

where

$$\begin{aligned} F(Y_t, Z_t) &= -r_t Y_t + Z_t E_\pi(M_{t+1}|\mathcal{F}_t) \\ &= -r_t Y_t + Z_t \begin{bmatrix} \pi - p \\ p - \pi \end{bmatrix} \end{aligned}$$

This is a special case of a BSDE.

Before giving general existence properties of BSDEs, we need the following.

Definition

If $Z_t^1 M_{t+1} = Z_t^2 M_{t+1}$ $\mathbb{P}$-a.s. for all t, then we write $Z^1 \sim_M Z^2$.

Note this is an equivalence relation for $Z_t(\omega) \in \mathbb{R}^{N\times 1}$.

Theorem

For any $\mathcal{F}_{t+1}$-measurable random variable $W \in \mathbb{R}$ with $E[W|\mathcal{F}_t] = 0$, there exists an $\mathcal{F}_t$-measurable Z_t with

$$W = Z_t M_{t+1}.$$

Proof.

Simple application of Doob-Dynkin Lemma. □

An Existence Theorem

Theorem

Suppose

(i) $F(\omega, t, y, z)$ *is invariant under equivalence* $\sim_M$ *in* z.

(ii) *For all* z*, the map*

$$y \mapsto y - F(\omega, t, y, z)$$

is a bijection $\mathbb{R} \to \mathbb{R}$.

Then a BSDE with driver F *has a unique solution for any terminal condition* Q *(all in* L^1*).*

Corollary

These conditions are necessary and sufficient.

Proof:

Let $Z_t \in \mathbb{R}^{N\times 1}$ solve

$$Z_t M_{t+1} = Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t].$$

Then, pathwise, let $Y_t \in \mathbb{R}$ solve

$$Y_t - F(\omega, t, Y_t, Z_t) = E[Y_{t+1}|\mathcal{F}_t]$$

for the above value of Z_t.
Then (Y_t, Z_t) solves the one step equation

$$Y_t - F(\omega, t, Y_t, Z_t) + Z_t M_{t+1} = Y_{t+1},$$

and the result follows by backwards induction.

A Comparison Theorem

- We wish to ensure that, when $Q^1 \geq Q^2$, the corresponding values $Y_t^1 \geq Y_t^2$ for all t.
- This will, (eventually), allow us to define a nonlinear expectation $\mathcal{E}$ and obtain the monotonicity and concavity assumptions.
- The key theorem here is the *Comparison Theorem*

Definition

We define $\mathbb{J}_t$, the set of possible jumps of X from time t to time $t+1$, by

$$\mathbb{J}_t := \{i : \mathbb{P}(X_{t+1} = e_i | \mathcal{F}_t) > 0)\}.$$

Comparison Theorem

Theorem

Consider two BSDEs with drivers F^1, F^2, terminal values Q^1, Q^2, etc... Suppose that, $\mathbb{P}$-a.s. for all t,

(i) $Q^1 \geq Q^2$

(ii) $F^1(\omega, t, Y_t^2, Z_t^2) \geq F^2(\omega, t, Y_t^2, Z_t^2)$

(iii) $F^1(\omega, t, Y_t^2, Z_t^1) - F^1(\omega, t, Y_t^2, Z_t^2)$
$$\geq \min_{j \in \mathbb{J}_t}\{(Z_t^1 - Z_t^2)(e_j - E[X_{t+1}|\mathcal{F}_t])\}.$$

(iv) *The map $y \mapsto y - F(\omega, t, y, Z_t^1)$ is strictly increasing in y.*

Then $Y_t^1 \geq Y_t^2$ $\mathbb{P}$-a.s. for all t.

A function F^1 satisfying (iii) and (iv) will be called *balanced*.

Proof:

Assume $Y^1_{t+1} - Y^2_{t+1} \geq 0$, then, omitting ω and t,

$$\begin{aligned}
&Y^1_t - Y^2_t - F^1(Y^1_t, Z^1_t) + F^1(Y^2_t, Z^1_t) \\
&\quad - [F^1(Y^2_t, Z^1_t) - F^1(Y^2_t, Z^2_t)] + (Z^1_t - Z^2_t)M_{t+1} \\
&= [Y^1_{t+1} - Y^2_{t+1}] + [F^1(Y^2_t, Z^2_t) - F^2(Y^2_t, Z^2_t)] \\
&\geq 0.
\end{aligned}$$

This must hold $\mathbb{P}$-a.s., so it holds under taking the $\mathcal{F}_t$-conditional essential minimum of all terms. Hence

$$Y^1_t - Y^2_t - F^1(Y^1_t, Z^1_t) + F^1(Y^2_t, Z^1_t) \geq 0$$

and then as $Y_t \mapsto Y_t - F(Y_t, Z^1_t)$ is strictly increasing, the result follows by induction.

Outline

Nonlinear Expectations

For some terminal time T, we define an '$\mathcal{F}_t$-consistent nonlinear expectation' $\mathcal{E}$ to be a family of operators

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t); t \leq T$$

with

1. (Monotonicity) If $Q^1 \geq Q^2$ $\mathbb{P}$-a.s.,

$$\mathcal{E}(Q^1|\mathcal{F}_t) \geq \mathcal{E}(Q^2|\mathcal{F}_t)$$

2. (Constants) For all $\mathcal{F}_t$-measurable Q,

$$\mathcal{E}(Q|\mathcal{F}_t) = Q$$

Nonlinear Expectations

3. (Recursivity) For $s \leq t$,

$$\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s)$$

4. (Zero-One law) For any $A \in \mathcal{F}_t$,

$$\mathcal{E}(I_A Q|\mathcal{F}_t) = I_A \mathcal{E}(Q|\mathcal{F}_t).$$

Nonlinear Expectations

Two other properties are desirable

5. (Translation invariance) For any $q \in L^2(\mathcal{F}_t)$,

$$\mathcal{E}(Q + q|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t) + q.$$

6. (Concavity) For any $\lambda \in [0, 1]$,

$$\mathcal{E}(\lambda Q^1 + (1 - \lambda)Q^2|\mathcal{F}_t) \geq \lambda\mathcal{E}(Q^1|\mathcal{F}_t) + (1 - \lambda)\mathcal{E}(Q^2|\mathcal{F}_t)$$

Nonlinear Expectations

There is a relation between nonlinear expectations and convex risk measures:

- If (1)-(6) are satisfied, then for each t,

  $$\rho_t(X) := -\mathcal{E}(X|\mathcal{F}_t)$$

  defines a dynamic convex risk measure. These risk measures are *time consistent*.
- For simplicity, this presentation will discuss nonlinear expectations.
- How could we construct such a family of operators?

BSDEs and Nonlinear Expectations

Theorem

The following statements are equivalent:

(i) $\mathcal{E}(\cdot|\mathcal{F}_t)$ *is an* $\mathcal{F}_t$*-consistent, translation invariant nonlinear expectation. (Axioms 1-5)*

(ii) *There is an* F *such that* $Y_t = \mathcal{E}(Q|\mathcal{F}_t)$ *solves a BSDE with driver* F *and terminal condition* Q*, where* F *is balanced, independent of* Y_t*, and* $F(\omega, t, Y_t, 0) = 0$ $\mathbb{P}$*-a.s. for all* t*.*

In this case,

$$F(\omega, t, y, z) = \mathcal{E}(zM_{t+1}|\mathcal{F}_t).$$

NB. This result holds for both scalar and vector valued nonlinear expectations.

Corollary

The nonlinear expectation $\mathcal{E}(\cdot|\mathcal{F}_t)$ has property '...' if and only if F has property '...' (in Z), where '...' is any of:

- *Concavity*
- *Positive homogeneity*
- *Linearity*
- *Invariance under addition of martingale terms orthogonal to a given process*
- *(Lipshitz) continuity (in L^1 norm)*
- *etc...*

Note, these statements are trivial, given the equivalence

$$F(\omega, t, Y_t, Z_t) = \mathcal{E}(Z_t M_{t+1}|\mathcal{F}_t).$$

Lemma

The following statements are equivalent:

- *H is a map $L^1(\mathbb{R}^m; \mathcal{F}_t) \to L^1(\mathbb{R}^n; \mathcal{F}_t)$ satisfying the zero-one law*

$$H(I_A X + I_{A^c} X') = I_A H(X) + I_{A^c} H(X')$$

  for all $A \in \mathcal{F}_t$.

- *There exists a $\mathcal{F}_t$-measurable function $F : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ such that*

$$H(X)(\omega) \equiv F(\omega, X(\omega)).$$

Proof.

Simply decompose Ω into events of the form $A = \{\omega : X = x\}$ and we see that $H(xI_\Omega) = F(\omega, x)$ is as desired. $\square$

Proof of representation.

($\Leftarrow$) A BSDE yields a nonlinear expectation in the usual way.
($\Rightarrow$) Define the functional $H(\omega, t, y, z) = \mathcal{E}(zM_{t+1}|\mathcal{F}_t)$. By regularity of $\mathcal{E}$, we match it with a function F by the previous lemma. Then, for $Y_t := \mathcal{E}(Y_{t+1}|\mathcal{F}_t)$ and $Z_t M_{t+1} := Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t]$, we have

$$\begin{aligned} Y_{t+1} &= E[Y_{t+1}|\mathcal{F}_t] + Z_t M_{t+1} \\ &= E[Y_{t+1}|\mathcal{F}_t] + \mathcal{E}(Z_t M_{t+1}|\mathcal{F}_t) - \mathcal{E}(Z_t M_{t+1}|\mathcal{F}_t) + Z_t M_{t+1} \\ &= \mathcal{E}(E[Y_{t+1}|\mathcal{F}_t] + Z_t M_{t+1}|\mathcal{F}_t) - \mathcal{E}(Z_t M_{t+1}|\mathcal{F}_t) + Z_t M_{t+1} \\ &= \mathcal{E}(Y_{t+1}|\mathcal{F}_t) - F(\omega, t, Y_t, Z_t) + Z_t M_{t+1} \\ &= Y_t - F(\omega, t, Y_t, Z_t) + Z_t M_{t+1} \end{aligned}$$

Uniqueness of F is easily verified. The fact F is balanced follows from the monotonicity of $\mathcal{E}$. □

An Example

Consider a two step world where X_t takes one of two values with equal probability.

Assume that Z_t is written in the form $Z_t = [z, -z]$, which is unique up to equivalence $\sim_{M_t}$. We consider the concave function

$$F(\omega, t, Y_t, Z_t) = \min_{\pi \in [0.1, 0.9]} \{2(\pi - 0.5)z + \gamma(\pi - 0.5)^2\},$$

where γ is a 'risk aversion' parameter (the smaller the value of γ, the more risk averse), which we shall set to $\gamma = 10$.

Other Results:

- One can show under what conditions a generic monotone map $L^2(\mathcal{F}_T) \to \mathbb{R}$ can be extended to an $\mathcal{F}_t$ consistent nonlinear expectation.
- It is also possible, in general, to determine under what conditions the driver F can be determined from the solutions Y_t, even when the comparison theorem and normalisation conditions do not hold.
- These results are significantly stronger than available in continuous time.

Outline

Infinitely many outcomes

- We now shall extend this theory to the case where we are still in discrete time, but infinitely many outcomes are possible.
- Clearly, as our martingale representation theorem (MRT) required a martingale in $\mathbb{R}^N$ for a process with N states, a discrete-time infinite-state situation will not have a finite-dimensional MRT (i.e. the multiplicity of the filtration is infinite).
- When $L^2(\mathcal{F}_T)$ is separable, we can obtain a countable dimensional MRT for square-integrable martingales.
- Here we will simply not use a representation.

Discrete-time, Infinite-state BSDEs

Consider an equation of the form

$$Y_t - \sum_{t \leq u < T} F(\omega, u, Y_u, Z_{u+1}) + \sum_{t \leq u < T} Z_{u+1} = Q$$

where

- $F : \Omega \times \{0, ..., T-1\} \times \mathbb{R}^K \times L^1(\mathcal{F}_t) \to \mathbb{R}^K$
- Y is adapted, and Z is a martingale difference process.
- Note: If we have a martingale representation theorem, then we can write $Z_{t+1} = \tilde{Z}_t M_{t+1}$, and so we obtain the earlier results.

$$Y_t - \sum_{t \leq u < T} F(\omega, u, Y_u, Z_{u+1}) + \sum_{t \leq u < T} Z_{u+1} = Q$$

- As F is now a functional on $L^1(\mathcal{F}_t)$, it does not distinguish a.s. equal processes.
- If $Y \mapsto \phi(Y) := Y - F(\omega, t, Y, Z_t)$ is a bijection with an integrable inverse (i.e. $\phi^{-1}(Y)$ is integrable for Y integrable), then for any integrable Q, we have a unique solution in L^1.
- We may also wish to allow F to take the value ∞ for some y, as this will help our representation results.

Many of the earlier results can be obtained in this context.

- The requirements of the comparison theorem are now in terms of essential infima, not minima.
- Translation invariant nonlinear expectations have the same representation.
- In the scalar case, general nonlinear evaluations have a BSDE representation, however this depends on obtaining solutions $F(\omega, t, Y_t, Z_{t+1}) = c_t$ where

$$Y_t = \mathcal{E}(Y_t + Z_{t+1} + c_t).$$

  We allow $F(\omega, t, Y_t, Z_{t+1}) = \infty$ whenever this does not have a solution.

Outline

BSDEs in general spaces

- Often, we wish to use discrete time BSDEs as a simplification of continuous time BSDEs.
- In these cases, we think of the discrete BSDE as a discretisation of the continuous time BSDE, and then allow our discretisation mesh to converge.
- This is not the only way discrete and continuous time can be related!
- We now consider a continuous time BSDE theory that allows the classical BSDE, and the discrete time theory, to be embedded as special cases.

Theorem (Davis & Varaiya 1974)

Let $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ *be a filtered probability space. Suppose* $L^2(\mathcal{F}_T)$ *is separable. Then there exists a sequence of martingales* $M^1, M^2 ...$ *such that any martingale N can be written as*

$$N_t = N_0 + \sum_{i=1}^{\infty} \int_{]0,t]} Z_u^i dM_u^i$$

for some predictable processes Z^i*, and*

$$\langle M^1 \rangle_t \succ \langle M^2 \rangle_t \succ ...$$

as measures on $\Omega \times [0, T]$*.*

Definition

For μ a fixed nonnegative Stieltjes measure with $\mathbb{P} \times \mu \succ \mathbb{P} \times \langle M^1 \rangle_t$, let $\| \cdot \|_{M_t}$ be the norm on infinite $\mathbb{R}^K$-valued sequences given by

$$\|(z^1, z^2, ...)\|^2_{M_t} = \sum_{i=0}^{\infty} \|z^i\|^2 \frac{d\langle M^i \rangle_t}{d(\mathbb{P} \times \mu_t)}.$$

We assume that such a measure μ exists.

BSDEs in general spaces

Consider an equation of the form:

$$Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, \mathbf{Z}_u) d\mu + \sum_{i=1}^{\infty} \int Z_u^i dM_u^i = Q$$

where

- $Q \in L^2(\mathcal{F}_T)$,
- $Y \in \mathbb{R}^K$ is adapted and $\sup_{t\in[0,T]}\{\|Y_t\|^2\} < \infty$,
- $\mathbf{Z}_t \equiv (Z^1, Z^2, ...)$ is a sequence of predictable $\mathbb{R}^K$-valued processes such that $\mathbf{Z} \in \mathcal{H}_M^2$, that is

$$E\left[\sum_i \int_{]0,T]} \|Z_t^i\|^2 d\langle M^i\rangle_t\right] = E\left[\int_{]0,T]} \|\mathbf{Z}_u\|_{M_u}^2 d\mu_t\right] < \infty$$

BSDEs in general spaces

$$Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, \mathbf{Z}_u) d\mu + \sum_{i=1}^{\infty} \int Z_u^i dM_u^i = Q$$

Also,

- μ is a deterministic Stieltjes measure on $[0, T]$. For simplicity, assume μ is nonnegative and $\mathbb{P} \times \mu \succ \langle M \rangle_t$
- F is a progressively measurable function such that $F(\omega, t, 0, \mathbf{0})$ is μ-square-integrable.

Existence result

Theorem

Suppose F is firmly Lipschitz, that is, there exists a constant c and a map $c_{(\cdot)} : [0, T] \to [0, c]$ such that

$$\|F(\omega, t, y, \mathbf{z}) - F(\omega, t, y', \mathbf{z}')\|^2 \leq c_t \|y - y'\|^2 + c\|\mathbf{z} - \mathbf{z}'\|^2_{M_t}$$

and

$$c_t(\Delta\mu_t)^2 < 1.$$

Then the BSDE has a unique solution, (up to indistinguishability if $d\mu \succ dt$).

- As the discrete time BSDE can be embedded in continuous time, and the necessary and sufficient condition for existence in discrete time is that $y \mapsto y - F(\omega, t, y, z)$ is a bijection, the classical requirement of Lipschitz continuity is clearly insufficient.
- On the other hand, if μ is continuous, then these assumptions are simply classical Lipschitz continuity.
- By the use of the Radon-Nikodym theorem for measures on $\Omega \times [0, T]$, the requirement that μ is deterministic, nonnegative and $\mathbb{P} \times \mu \succ \langle M^1 \rangle_t$ is a very flexible one, as exceptions can be instead incorporated into F.

- Proving this theorem requires a variant of a backwards Grönwall inequality for Doléans-Dade exponentials of finite-variation processes.
- The existence proof is otherwise fairly standard, using the contraction mapping theorem for a local result on sets in $[0, T]$ of μ-measure < 1 assuming $c_t \Delta\mu_t < 1$
- An inductive argument with a measure change extends this to all times assuming $c_t(\Delta\mu_t)^2 < 1$.
- There is no particular reason to assume $T < \infty$, provided $\mu_T < \infty$.

Balanced drivers

Consider the scalar case only.

Definition

Let F be such that for $(Y^1, \mathbf{Z}^1), (Y^2, \mathbf{Z}^2)$ the solutions to two BSDEs, we have that

$$-\int_{]0,t]} [F(\omega, u, Y^1_{u-}, \mathbf{Z}^1_u) - F(\omega, u, Y^1_{u-}, \mathbf{Z}^2_u)] d\mu_u$$
$$+ \sum_i \int_{]0,t]} [(Z^1)^i_u - (Z^2)^i_u] dM^i_u$$

has an equivalent martingale measure. Then F shall be called balanced.

Comparison Theorem

Consider the scalar case only.

Theorem

Let $(Y^1, \mathbf{Z}^1)$ and $(Y^2, \mathbf{Z}^2)$ be the solutions to two BSDEs with drivers F^1, F^2 and terminal conditions Q^1, Q^2. Then if

- $Q^1 \geq Q^2$ *a.s.*
- $F^1(\omega, t, Y^2_{t-}, Z^2) \geq F^2(\omega, t, Y^2_{t-}, Z^2_t)$ $\mu \times \mathbb{P}$*-a.s. and*
- F^1 *is balanced*

It follows that $Y^1_t \geq Y^2_t$ for all t. The strict comparison also applies.

- These conditions are the natural extension of the requirements in discrete time, which can be shown to be necessary for the general result to hold.
- As the comparison theorem is the non-linear version of a no-Arbitrage result, it is natural to think of it in terms of equivalent-martingale-measures.
- This also indicates that, perhaps with generalisation to local- or σ-martingales, it may be the most general condition to use.
- The various classical examples of the comparison theorem can all be seen to be special cases of this requirement.

- With this theory, we can show that these BSDEs generate nonlinear expectations and evaluations in these spaces.
- It is plausible that all nonlinear expectations, satisfying appropriate boundedness conditions, can be obtained in this way.
- These BSDEs unify much of the discrete and continuous time theories of BSDEs in a single construction, and require very weak assumptions on the underlying probability space.

Outline

Conclusions

- The theory of BSDEs has many extensions beyond the classical case.
- We can obtain many of the classical results in these new contexts, and in discrete time, the proofs are often simpler.
- Discontinuities in the filtration can be incorporated in BSDE theory without much difficulty, but existence results require stronger assumptions than when the filtration is continuous.
- When discontinuities are present, we can still obtain a comparison theorem, but a further assumption is required. This assumption can be seen in the context of the existence of equivalent martingale measures, making a strong link with no-arbitrage results in mathematical finance.