# Penalty Methods for the Numerical Solution of Hamilton-Jacobi-Bellman Equations in Finance

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# Structure of this Talk

1. Motivation / Examples of Nonlinear Equations in Finance
2. Literature Review / Existing Methods
3. Notes on Policy Iteration
4. Penalty Approach
   - 4.1 Discretise the Continuous Equation
   - 4.2 Penalise the Nonlinear Discrete System
   - 4.3 Iteratively Solve the Penalised Problem
5. Numerical Results (and Pictures)
6. Conclusion

# Linear or nonlinear

- The majority of pricing equations used in financial engineering practice are linear.
- The Black-Scholes PDE and its variants are linear.
- If option $A$ has payoff $G_A$ and value $V_A$ and option $B$ has payoff $G_B$ and value $V_B$, then an option with payoff $G_A + G_B$ has value $V_A + V_B$.

Why should this not be so?

- **American options**: non-linearity through early exercise right.
- **Incomplete markets**: An agents' risk preferences become relevant and are typically non-linear.
- **Uncertain parameters**: Financial models typically contain parameters which are not known precisely. This leads to nonlinear upper and lower price bounds.
- Transaction costs, illiquid markets, large trader price impact,...

# HJB Equations in Finance

## Early Exercise Options

Let $(B_t)$ and $(S_t)$ represent bond and stock, satisfying

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t \end{aligned}$$

where $W_t$ is a Brownian motion.
The value function $V$ of an American option with *payoff* $\psi$ satisfies

$$\begin{aligned} \min(\mathcal{L}V, V - \psi) =& 0, \\ V(\cdot, T) =& \psi, \end{aligned}$$

where

$$-\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV.$$

# HJB Equations in Finance

## Uncertain volatility

If there is uncertainty about model parameters, as there is no complete probabilistic underpinning, we can only investigate best/worst case scenarios.

- Assume $dS_t/S_t = \mu dt + \sigma_t dW_t$ with $\sigma_t \in \Sigma = [\underline{\sigma}, \overline{\sigma}]$.
- Set up delta-hedged portfolio $\Pi = V - \Delta S$ with $\Delta = \frac{\partial V}{\partial S}$,

$$d\Pi_t = (\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 V}{\partial S^2})dt.$$

- For sub-replication, $\min_{\sigma \in \Sigma} d\Pi_t = r\Pi_t$.
- Inserting leads to the pricing PDE

$$\frac{\partial V}{\partial t} + \min_{\sigma \in \Sigma} \left( \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) + rS\frac{\partial V}{\partial S} - rV = 0.$$

- Similar for upper bound with “min” replaced by “max”.

See Avellaneda, Levy, Para (1995), Lyons (1995).

# HJB Equations in Option Pricing

## Stock Borrowing Fees and Unequal Borrowing/Lending Rates

Introducing market frictions.

- Let $r_b \geq r_l \geq r_f \geq 0$ denote borrowing, lending and stock borrowing fees, respectively.

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## Stock Borrowing Fees and Unequal Borrowing/Lending Rates

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- Let $r_b \geq r_l \geq r_f \geq 0$ denote borrowing, lending and stock borrowing fees, respectively.
- For $r$, $q$, $\sigma > 0$, we introduce the Black-Scholes operator

$$\mathcal{L}_{BS}^{r,q,\sigma} V := \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV.$$

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- To price a short position in a European option, we have to solve (cf. Amadori,'03)

$$\max\left\{\mathcal{L}_{BS}^{r,q,\sigma} V : (r,q) \in \mathbf{U}\right\} = 0,$$

where $\mathbf{U} := \{(r_l, 0), (r_b, 0), (r_l, r_f), (r_b, r_b - r_l + r_f)\}$.

# HJB Equations in Finance

## Stochastic optimisation

Merton problem:

- Black-Scholes set-up $dS_t/S_t = \mu dt + \sigma dW_t$, put $\pi_t$ in stock
- The wealth process $X$ follows, with $\lambda = (\mu - r)/\sigma$,

$$dX_t = \sigma \pi_t (\lambda dt + dW_t).$$

- Maximise the terminal utility $U(X_T)$, then the value function

$$u(x,t) = \sup_{\pi} E(U(X_T) | X_t = x)$$

satisfies the *HJB equation*

$$\begin{aligned} \sup_{\pi} \left( \frac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2 \pi^2 \frac{\partial^2 u}{\partial x^2} + \sigma \lambda \pi \frac{\partial u}{\partial x} \right) &= 0, \\ u(x,T) &= U(x). \end{aligned}$$

# HJB Equations in Finance

## Portfolio Optimisation in an Incomplete Market

Let $(B_t)$ and $(S_t)$ bond and stock, with $dB_t = rB_t dt$,

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma(Y_t) S_t dW_t^1 \\ \text{and} \quad dY_t &= b(Y_t)dt + a(Y_t)dW_t^2, \end{aligned}$$

where $W_t^1$, $W_t^2$ are two Brownian motions correlated by $\varrho$.

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The investor maximises terminal utility $U(x, y) = \frac{1}{\gamma} x^\gamma h(y)$, where $x$ is the current wealth.

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The value function is (Zariphopoulou,'01) $v(x, y, t) = \frac{x^\gamma}{\gamma} V(y, t)$, where $V$ solves

$$\frac{1}{\gamma}\left[V_t + \frac{1}{2}a^2(y)V_{yy} + b(y)V_y\right] + rV$$
$$+\max_{\pi\in\mathbb{R}}\left[\frac{1}{2}(\gamma-1)\sigma^2(y)\pi^2 V + \varrho\sigma(y)a(y)\pi V_y + (\mu-r)\pi V\right] = 0.$$

# HJB Equations in Finance

## Early Exercise in an Incomplete Market

Consider traded and non-traded asset $(S_t)$ and $(Y_t)$, respectively,

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^1$$
$$\text{and} \quad dY_t = b(Y_t) \, dt + a(Y_t) \, dW_t^2 \,,$$

where $W_t^1$, $W_t^2$ are two Brownian motions correlated by $\varrho$, $r = 0$.

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Assume the investor's utility function $U(x) = -e^{-\gamma x}$, $x$ wealth.

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where $W_t^1$, $W_t^2$ are two Brownian motions correlated by $\varrho$, $r = 0$.

Assume the investor's utility function $U(x) = -e^{-\gamma x}$, $x$ wealth.

The buyer's early exercise indifference price of an American option with payoff $P$ on $(Y_t)$ solves (Oberman & Zariphopoulou,'03)

$$\min\left\{ \max_{u\in\mathbb{R}} \mathcal{L}_u^b \psi\ , \psi - P(y)\right\} = 0,$$

$$\text{where} \quad \begin{aligned} \mathcal{L}_u^b\psi := -\psi_t - \frac{1}{2}a^2(y)\psi_{yy} - \left(b(y) - \rho\frac{\mu}{\sigma}a(y)\right)\psi_y \\ + \frac{1}{2}\gamma(1-\rho^2)a^2(y)(2u\psi_y - u^2). \end{aligned}$$

# Existing Work on the Topic

HJB equations arise when applying Bellman's principle of optimality to problems of stochastic optimal control.

Markov chain approximation:

- H. J. Kushner, Numerical Methods for Stochastic Control Problems in Continuous Time, SIAM Journal on Control and Optimization, 1990.
- H. J. Kushner, P. G. Dupuis, Numerical methods for Stochastic Control Problems in Continuous Time, 2001.

- W. H. Fleming, H. M. Soner, Controlled Markov Processes and Viscosity Solutions, 2005.

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Finite Differences:

- G. Barles, Convergence of Numerical Schemes for Degenerate Parabolic Equations arising in Finance, Numerical Methods in Finance, 1997.
- P. A. Forsyth, G. Labahn, Numerical Methods for Controlled HJB PDEs in Finance, Journal of Computational Finance, 2007.

# Nonlinear Equations in Finance

## Considering an HJB Equation and an HJB Obstacle Problem

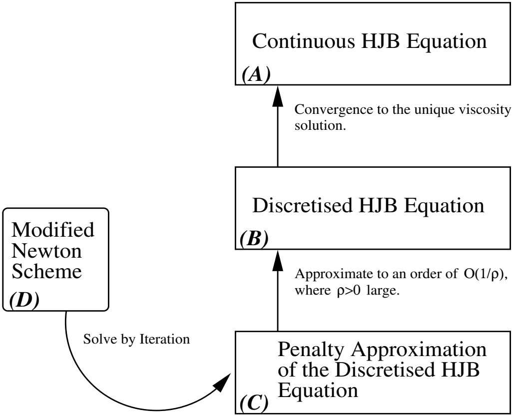
**Problem 1:** Let $\mathbf{U} \subset \mathbb{R}$ be a compact interval. Let $\mathcal{L}_u$, $u \in \mathbf{U}$, be a family of linear differential operators on $\Omega \subset \mathbb{R}^n$. Find a function $V : \Omega \to \mathbb{R}$ such that

$$\inf_{u \in \mathbf{U}} \left[ \mathcal{L}_u V \right] = 0.$$

**Problem 2:** Let $\tilde{\mathcal{L}}$ be an additional linear differential operator. Find a function $W : \Omega \to \mathbb{R}$ such that

$$\inf \left\{ \sup_{u \in \mathbf{U}} \left[ \mathcal{L}_u W \right], \tilde{\mathcal{L}} W \right\} = 0.$$

# Overview of our Approach



# Review linear PDEs

Recall linear parabolic (= 2d degenerate elliptic) PDEs,

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + r(x,t)u = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}.$$

Usually unique classical (i.e. sufficiently differentiable) solutions. Analysis of difference schemes then by checking

- consistency (via Taylor expansion);
- stability (via discrete maximum principles or Fourier analysis).

# Nonlinear PDEs

- We consider non-linear (second order) PDEs in the form

$$\frac{\partial u}{\partial t} + \Phi\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) = 0,$$

which are backward parabolic if

$$p \le q \quad \Rightarrow \quad \Phi(x, t, u, s, p) \le \Phi(x, t, u, s, q).$$

- In particular, we have in mind equations of the form

$$\frac{\partial u}{\partial t} + \sup_p \left(\mu(x, t; p)\frac{\partial u}{\partial x} - r(x, t; p)u + \frac{1}{2}\sigma^2(x, t; p)\frac{\partial^2 u}{\partial x^2}\right) = 0.$$

- There is not necessarily a solution in the classical sense. This gives rise to the concept of (continuous) *viscosity* solution, which is the solution (i.e. value function) to the underlying control problem.

# Finite differences for nonlinear PDEs

Define a finite difference scheme

$$\delta_t u + \Phi\left(x, t, u, \delta_x u, \delta_x^2 u\right) = 0,$$

or in the form

$$\phi(x_n, t_m, \Delta x, \Delta t, \{u_{n-1}^m, u_{n+1}^m, u_{n-1}^{m+1}, u_n^{m+1}, u_{n+1}^{m+1}\}, u_n^m) = 0.$$

Analysis via the following:

- Consistency: For the true solution $u$,

$$\delta_t u + \Phi\left(x, t, u, \delta_x u, \delta_x^2 u\right) \to 0 \quad \text{for} \quad \Delta t, \Delta x \to 0.$$

- Stability: $\max |u| \leq c$.
- Monotonicity:

$$\phi(x, t, \Delta x, \Delta t, \{\ldots\}, u) \geq \phi(x, t, \Delta x, \Delta t, \{\ldots\}, v) \quad \text{if} \quad u \geq v.$$

See Barles (1991).

# Discretised equations

- For an explicit scheme, it is straightforward to solve

$$\frac{u_n^m - u_n^{m-1}}{\Delta t} + \Phi\left(x, t, u_n^m, \delta_x u_n^m, \delta_x^2 u_n^m\right) = 0$$

backwards in time from terminal values $u_n^M$.

- There is usually a stability constraint $\Delta t = O(\Delta x^2)$.
- Implicit schemes are typically unconditionally stable (monotone), and allow higher order for smooth solutions.
- Implicit schemes require the solution of a large non-linear discrete system of equations, of the type

$$\frac{u_n^m - u_n^{m-1}}{\Delta t} + \Phi\left(x, t, u_n^{m-1}, \delta_x u_n^{m-1}, \delta_x^2 u_n^{m-1}\right) = 0,$$

or, in matrix form using control formulation, row-wise

$$\sup_u \left(A_u V - b_u\right) = 0, \quad A_u \in \mathbb{R}^{N \times N}, V, b_u \in \mathbb{R}^N.$$

# Literature


- H.J. Kushner and P. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer, 2001.
- G. Barles, Convergence of numerical schemes for degenerate parabolic equations arising in finance. In L. C. G. Rogers and D. Talay (Eds.), *Numerical Methods in Finance*, pp. 1-21, CUP, Cambridge, 1991.
- A. Oberman and T. Zariphopoulou, Pricing early exercise contracts in incomplete markets. Computational Management Science, 1:75–107, 2003.
- P. A. Forsyth and G. Labahn. Numerical methods for controlled Hamilton-Jacobi-Bellman PDEs in Finance. The Journal of Computational Finance, 11(2):1–44, 2007.

# Solving HJB Equations Numerically

## How Simple Finite Differences Produce Nontrivial Discrete Problems

$$\inf_{u\in\mathbf{U}}\left[\mathcal{L}_u V\right]=0 \qquad \inf\left\{\sup_{u\in\mathbf{U}}\left[\mathcal{L}_u W\right],\tilde{\mathcal{L}}W\right\}=0$$

**Discr. Prob. 1:** Find $x\in\mathbb{R}^N$ such that

$$\min_{u\in\mathbf{U}}\left[A_u x-b_u\right]=0.$$

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**Discr. Prob. 2:** Find $z \in \mathbb{R}^N$ such that

$$\min\left\{ \max_{u\in\mathbf{U}} \left[A_u z - b_u\right], \tilde{A} z - \tilde{b} \right\} = 0.$$

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$$\min\left\{ \max_{u\in\mathbf{U}} \left[A_u z - b_u\right], \tilde{A} z - \tilde{b} \right\} = 0.$$

We assume that

- $b_u \in \mathbb{R}^N$ and $A_u \in \mathbb{R}^{N\times N}$, $u \in \mathbf{U}$, are vectors and monotone discretisation matrices, respectively,
- and $u \mapsto b_u$ and $u \mapsto A_u$ are continuous functions on $\mathbf{U}$.

# The Method of Policy Iteration

## Briefly Explained

We want to solve

$$\min_{u \in \mathbf{U}} \left[ A_u x - b_u \right] = 0.$$

**Algorithm:** (Forsyth & Labahn,'07) Pick a starting value $x^0$. For given $x^n$, find $x^{n+1}$ such that

$$A_{u(x^n)} x^{n+1} = b_{u(x^n)},$$

where $A_{u(x^n)} x^n - b_{u(x^n)} = \min_{u \in \mathbf{U}} \left[ A_u x^n - b_u \right]$.

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where $A_{u(x^n)} x^n - b_{u(x^n)} = \min_{u \in \mathbf{U}} \left[ A_u x^n - b_u \right]$.

- The method of policy iteration generates a globally convergent monotone sequence.
- It has been shown by Bokanowski *et al.* ('09) how to apply policy iteration to HJB obstacle problems.

# Example: American options

- Let $A \in \mathbb{R}^{N \times N}$ be an M-matrix, and let $b$, $c \in \mathbb{R}^N$ be vectors.
- Find $x \in \mathbb{R}^N$ such that

$$
\begin{aligned}
Ax &\geq b, \\
x &\geq c \\
\text{and} \quad (Ax - b)_i \cdot (x - c)_i &= 0, \quad 1 \leq i \leq N.
\end{aligned}
$$

- In particular, an M-matrix $Z$ is non-singular with $Z^{-1} \geq 0$, i.e. every element of $Z^{-1}$ is non-negative.
- Matrices of this form arise naturally from most discretisation schemes for partial differential equations.

# Example: American options

- Denote $I_N$ to denote the identity matrix in $\mathbb{R}^{N \times N}$.
- Let $x^0 \in \mathbb{R}^N$.
- For $x^n$ given, let $A^n \in \mathbb{R}^{N \times N}$ and $b^n \in \mathbb{R}^N$ be such that
- $(A^n)_i \in \{(A)_i, (I_N)_i\}$ and $(b^n)_i \in \{(b)_i, (c)_i\}$ for $1 \leq i \leq N$,
- and
$$(A^n x^n - b^n)_i = \min\left\{(Ax^n - b)_i, (x^n - c)_i\right\}.$$
- Find $x^{n+1} \in \mathbb{R}^N$ such that
$$A^n x^{n+1} = b^n.$$

# Performance policy iteration

| $r$ | $\sigma$ | T | K | $S_{max}$ | tol | $\rho$ |
|---|---|---|---|---|---|---|
| 0.05 | 0.4 | 1 | 100 | 600 | 1e-08 | 1e08 |

Table: The parameters used for the numerical computations.

| Policy Iteration, $M = N$ | 100 | 200 | 400 | 800 |
|---|---|---|---|---|
| Fully Implicit, $\varnothing$ Iterations | 1.05 | 1.07 | 1.07 | 1.07 |
| C.N. , $\varnothing$ Iterations | 1.02 | 1.03 | 1.04 | 1.04 |

Table: The average number of policy iterations required per time step when setting $M = N$ appears to be constant, for fully implicit as well as Crank-Nicolson (C.N.) time stepping. Hence, in practice, we can price American options in $O(N^2)$, which is the same complexity as for European options.

# Performance policy iteration

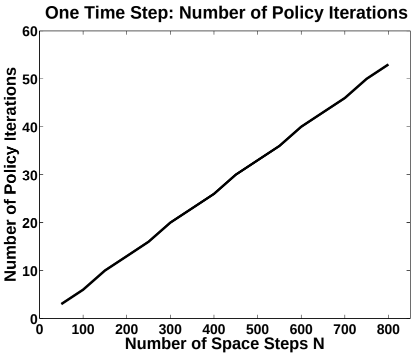


Figure: We fix $M = 1$, solving exactly one discrete LCP, and we consider different grid sizes $N$. We see that the number of policy iterations required for solving the discrete LCP is clearly linear in the matrix dimension $N$.

# The Idea of Penalisation

## Approximating one Nonlinear Problem by Another

In American option pricing, consider

$$\begin{aligned} \min\{\mathcal{L}V, V-\psi\} &= 0, \\ \mathcal{L}V_\rho - \rho\max\{\psi - V_\rho, 0\} &= 0, \end{aligned}$$

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and

$$\begin{aligned} \min\{Ax-b, x-c\} &= 0, \\ Ax_\rho - b - \rho \max\{c - x_\rho, 0\} &= 0. \end{aligned}$$

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and

$$\min\{Ax-b, x-c\} = 0,$$
$$Ax_\rho - b - \rho \max\{c - x_\rho\,, 0\} = 0.$$

For $\rho > 0$ large, we have the estimates

$$\|V - V_\rho\|_{L^2(0,T;H_0^1(\Omega))} \leq \frac{C}{\sqrt{\rho}} \quad \text{and} \quad \|x - x_\rho\|_\infty \leq \frac{C}{\rho}.$$

[cf. Bensoussan,Lions,'82, Forsyth,Vetzal,'02]

# Discrete LCP

- Recall the finite difference approximation for a European option with the $\theta$-method, in the form

$$KV^m = F$$

  in each time-step.

- The corresponding *linear complementarity problem* is

$$\begin{aligned} KV^m - F &\geq 0 \\ V^m &\geq G \\ (KV^m - F)\cdot(V^m - G) &= 0 \end{aligned}$$

  where both $\geq$ and $\cdot$ are understood to be elementwise.

- The penalty equation is

$$KV^m = F + \rho \max(G - V^m, 0).$$

# Convergence

- Assume standard conditions for maximum norm stability of the $\theta$-scheme.
- Then the finite difference solution of the penalty equation for the American put satisfies

$$
\begin{aligned}
KV^m - F &\geq 0 & & \\
V_n^m &\geq G_n - \epsilon & & 0 \leq n \leq N \\
(KV^m - F)_n = 0 \ \vee \ V_n^m - G_n &\leq \epsilon & & 0 \leq n \leq N
\end{aligned}
$$

- Here

$$\epsilon \leq \frac{1}{\rho}\left(c_0 + c_1 \frac{\Delta t}{\Delta S}\right)$$

  and $c_0$, $c_1$ independent of $\rho$, $\Delta t$, and $\Delta S$, i.e. can be made arbitrarily small by making $\rho$ sufficiently large.
- It follows that $V(\epsilon) \to V$ the solution (and of $O(\epsilon)$).

# Newton method

- To solve the non-linear penalised equation, write as

$$KV^m + \rho D(V^m)V^m = F + \rho D(V^m)G$$

with a diagonal matrix $D$ which has $D_{nn} = 1$ if $V_n^m < G_n$ and $D_{nn} = 0$ otherwise.

- Define

$$\frac{\partial\left(D_{nn}(V^m)(V_n^m - G_n)\right)}{\partial V_n^m} = \begin{cases} \rho & \text{if } \; V_n^m < G_n \\ 0 & \text{else} \end{cases}$$

to obtain the Newton iteration

$$KV^{m,k+1} + \rho D(V^{m,k})V^{m,k+1} = F + \rho D(V^{m,k})G.$$

- An obvious starting point is

$$V^{m,0} = V^{m-1}.$$

# Convergence

A suitable termination criterion is

$$\max_n \frac{|V_n^{m,k} - V_n^{m,k-1}|}{\max(1, V_n^{m,k})} \leq \delta \quad \vee \quad D(V^{m,k}) = D(V^{m,k-1})$$

with $\delta$ small. The latter implies that the *exercise region* does not change.

The iteration then has the following properties:

- It converges to the (unique) solution for any initial value.
- $V^{m,k+1} \geq V^{m,k}$ for $k \geq 1$ (monotonicity).
- The iteration terminates in a finite number of steps.

# Performance comparison

| *PSOR* | Max Iterations | ∅ Iterations | Runtime | $w_R^*$ |
|---|---|---|---|---|
| $M, N = 200$ | 3 | 2.17 | 0.23s | 1.050 |
| $M, N = 400$ | 5 | 2.67 | 1.07s | 1.075 |
| $M, N = 800$ | 5 | 3.15 | 4.88s | 1.175 |
| $M = 50, N = 800$ | 28 | 17.00 | 1.61s | 1.650 |
| $M = 800, N = 50$ | 1 | 1.00 | 0.14s | 1.000 |
| *Penalty Method* | Max Iterations | ∅ Iterations | Runtime | - |
| $M, N = 200$ | 2 | 1.06 | 0.04s | - |
| $M, N = 400$ | 2 | 1.06 | 0.08s | - |
| $M, N = 800$ | 3 | 1.07 | 0.28s | - |
| $M = 50, N = 800$ | 5 | 1.82 | 0.03s | - |
| $M = 800, N = 50$ | 2 | 1.00 | 0.07s | - |
| *Policy Iteration* | Max Iterations | ∅ Iterations | Runtime | - |
| $M, N = 200$ | 3 | 1.07 | 0.03s | - |
| $M, N = 400$ | 4 | 1.06 | 0.09s | - |
| $M, N = 800$ | 6 | 1.07 | 0.29s | - |
| $M = 50, N = 800$ | 18 | 2.08 | 0.04s | - |
| $M = 800, N = 50$ | 2 | 1.00 | 0.07s | - |

Table: Our computational results for different numbers of time and space steps, denoted by $M$ and $N$, respectively. We see that, for all chosen grid sizes, policy iteration and penalisation perform virtually identically and clearly outperform PSOR.

# Penalising a Discrete HJB Equation

## Separating one Operator and Penalising all Others

We approximate

$$\min_{u \in \mathbf{U}} \left[ A_u x - b_u \right] = 0$$

by

$$A_{u_0} x_\rho - b_{u_0} - \rho \max_{u \in \mathbf{U}} \left[ \max\{ b_u - A_u x_\rho \, , 0\} \right] = 0,$$

where $u_0 \in \mathbf{U}$.

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where $u_0 \in \mathbf{U}$.

For $\rho > 0$ large, we obtain the estimate

$$\| x - x_\rho \|_\infty \leq \frac{C}{\rho}.$$

# Penalising a Discrete HJB Obstacle Problem

## Separating the "max"-term and Penalising the Obstacle

We approximate

$$\min\left\{\max_{u\in\mathbf{U}}\left[A_u z - b_u\right], \tilde{A}z - \tilde{b}\right\} = 0$$

by

$$\max_{u\in\mathbf{U}}\left[A_u z_\rho - b_u\right] - \rho\max\left\{\tilde{b} - \tilde{A}z_\rho, 0\right\} = 0.$$

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by

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For $\rho > 0$ large, we obtain the estimate

$$\|z - z_\rho\|_\infty \leq \frac{C}{\rho}.$$

# Penalising the Nonlinear Equations

## What have we gained?

Since the penalised equations are nonlinear themselves, what have we gained?

# Penalising the Nonlinear Equations

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Since the penalised equations are nonlinear themselves, what have we gained?

$\Rightarrow$ If there is an easy way to solve the penalised discrete problems, then we can build two self-contained numerical schemes.

# An Iterative Method for the HJB Equation

## Using a Modification of Newton's Method for Solving the Penalised Equations

$$G^{HJB}(y) := (A_{u_0} y - b_{u_0}) - \rho \max_{u \in \mathbf{U}} \left[ \max\{b_u - A_u y, 0\} \right], \quad y \in \mathbb{R}^N,$$

$$J_G^{HJB}(y) := A_{u_0} + \rho \max_{u \in \mathbf{U}} \left[ A_u^y \right], \quad y \in \mathbb{R}^N.$$

**Algorithm:** Let $x^0 \in \mathbb{R}^N$. Then, for known $x^n$, $n \geq 0$, find $x^{n+1}$ such that

$$J_G^{HJB}(x^n)(x^{n+1} - x^n) = -G^{HJB}(x^n).$$

# An Iterative Method for the HJB Equation

## Using a Modification of Newton's Method for Solving the Penalised Equations

$$G^{HJB}(y) := (A_{u_0}\, y - b_{u_0}) - \rho \max_{u \in \mathbf{U}} \big[ \max\{b_u - A_u y, 0\} \big], \quad y \in \mathbb{R}^N,$$

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**Algorithm:** Let $x^0 \in \mathbb{R}^N$. Then, for known $x^n$, $n \geq 0$, find $x^{n+1}$ such that

$$J_G^{HJB}(x^n)(x^{n+1} - x^n) = -G^{HJB}(x^n).$$

$\Rightarrow$ It is $x^{n+1} \geq x^n$, $n \geq 1$, and, independently of $x_0$, the algorithm converges to a limit $x^*$ which solves $G^{HJB}(x^*) = 0$.

[For American options, a similar algorithm has been used by Forsyth and Vetzal,'02, Ito and Kunisch, '03.]

# An Iterative Method for the HJB Obstacle Problem

## Using a Modification of Newton's Method for Solving the Penalised Equations

$$G^{Obst}(y) := \max_{u \in \mathbf{U}} \left[ A_u y - b_u \right] - \rho \max \left\{ \tilde{b} - \tilde{A}\, y, 0 \right\}, \quad y \in \mathbb{R}^N,$$

$$J_G^{Obst}(y) := \max_{u \in \mathbf{U}} \left[ A_u^y \right] + \rho\, \tilde{A}^{y,+}, \quad y \in \mathbb{R}^N.$$

**Algorithm:** Let $z^0 \in \mathbb{R}^N$. Then, for known $z^n$, $n \geq 0$, find $z^{n+1}$ such that

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# An Iterative Method for the HJB Obstacle Problem

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**Algorithm:** Let $z^0 \in \mathbb{R}^N$. Then, for known $z^n$, $n \geq 0$, find $z^{n+1}$ such that

$$J_G^{Obst}(z^n)(z^{n+1} - z^n) = -G^{Obst}(z^n).$$

$\Rightarrow$ If $\tilde{A} = I_N$, then there exists a neighbourhood $\mathcal{B}$ of the solution $z^*$ such that, for any starting value $z^0 \in \mathcal{B}$, $(z^n)_{n \geq 0}$ remains in $\mathcal{B}$ and converges to $z^*$ at a quadratic rate.

# Numerical Solution of an HJB Equation

## Putting It All Together

1. We want to find a function $V$ such that

$$\min_{u \in \mathbf{U}} \left[ \mathcal{L}_u V \right] = 0.$$

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3. We approximate $x$ to an order $O(1/\rho)$ by $x_\rho$ satisfying
$$A_{u_0} x_\rho - b_{u_0} - \rho \max_{u \in \mathbf{U}} \left[ \max\{ b_u - A_u x_\rho \, , 0 \} \right] = 0,$$

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4. We use an iterative scheme to solve for $x_\rho$.

$\rightsquigarrow$ *Proceed analogously for the HJB Obstacle Problem.*

# HJB Equations in Option Pricing

## Stock Borrowing Fees and Unequal Borrowing/Lending Rates

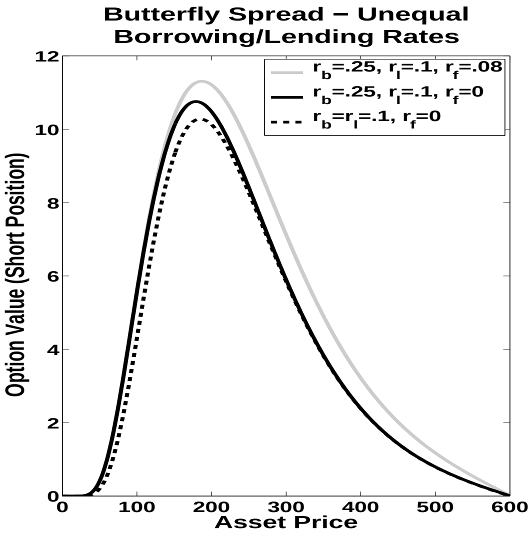
For $r,\ q,\ \sigma > 0$, we introduce the Black-Scholes Operator

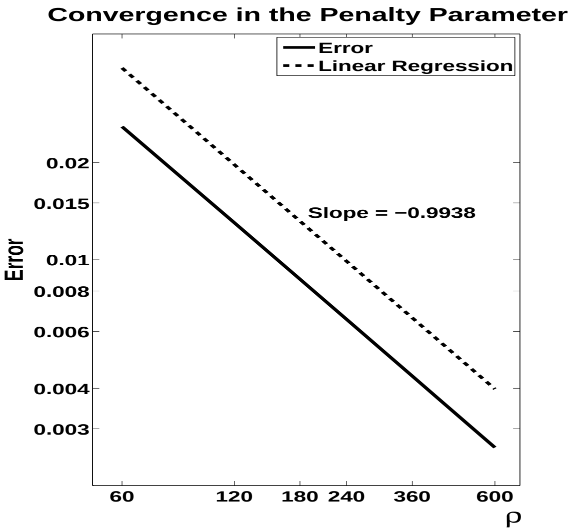
$$\mathcal{L}_{BS}^{r,q,\sigma} V := \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV.$$

Let $r_b \geq r_l \geq r_f \geq 0$ denote borrowing, lending and stock borrowing fees, respectively. To price a short position in a European option, we have to solve (cf. Amadori,'03)

$$\max\left\{\mathcal{L}_{BS}^{r,q,\sigma} V : (r,q) \in \mathbf{U}\right\} = 0,$$

where $\mathbf{U} := \{(r_l\,,0),(r_b\,,0),(r_l\,,r_f),(r_b\,,r_b - r_l + r_f)\}$.


Butterfly Spread – Unequal Borrowing/Lending Rates
Option Value (Short Position)
Asset Price
$r_b$=.25, $r_l$=.1, $r_f$=.08
$r_b$=.25, $r_l$=.1, $r_f$=0
$r_b$=$r_l$=.1, $r_f$=0
0
2
4
6
8
10
12
0
100
200
300
400
500
600


Convergence in the Penalty Parameter
Error
Linear Regression
Slope = −0.9938
Error
0.02
0.015
0.01
0.008
0.006
0.004
0.003
60
120
180
240
360
600
ρ

# Required Iterations

| *Policy Iteration* | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
|---|---|---|---|---|
| $M,\ N = 400$ | 90.50% | 9.50% | - | - |
| $M,\ N = 1000$ | 91.20% | 8.80% | - | - |
| $M = 900,\ N = 30$ | 99.67% | 0.33% | - | - |
| $M = 30,\ N = 900$ | - | 96.67% | 3.33% | - |
| *Penalty Method* $(\rho = 4e03)$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
| $M,\ N = 400$ | - | - | 78.75% | 21.25% |
| $M,\ N = 1000$ | - | - | 78.40% | 21.60% |
| $M = 900,\ N = 30$ | - | - | 92.56% | 7.44% |
| $M = 30,\ N = 900$ | - | - | 66.67% | 33.33% |
| *Penalty Method* $(\rho = 1e06)$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
| $M,\ N = 400$ | - | - | 79.00% | 21.00% |
| $M,\ N = 1000$ | - | - | 78.20% | 21.80% |
| $M = 900,\ N = 30$ | - | - | 92.44% | 7.56% |
| $M = 30,\ N = 900$ | - | - | 70.00% | 30.00% |

# Computational Time

| Grid Size | Policy | Penalty ($\rho = 4e03$) | Penalty ($\rho = 1e06$) |
|---|---|---|---|
| $M, N = 400$ | $0.216s$ | $0.733s$ | $0.740s$ |
| $M, N = 1000$ | $1.148s$ | $4.021s$ | $4.093s$ |
| $M = 900, N = 30$ | $0.142s$ | $0.473s$ | $0.475s$ |
| $M = 30, N = 900$ | $0.0624s$ | $0.116s$ | $0.121s$ |

# HJB Equations in Option Pricing

## Portfolio Optimisation in an Incomplete Market

Let $(B_t)$ and $(S_t)$ bond and stock, with $dB_t = rB_t dt$,

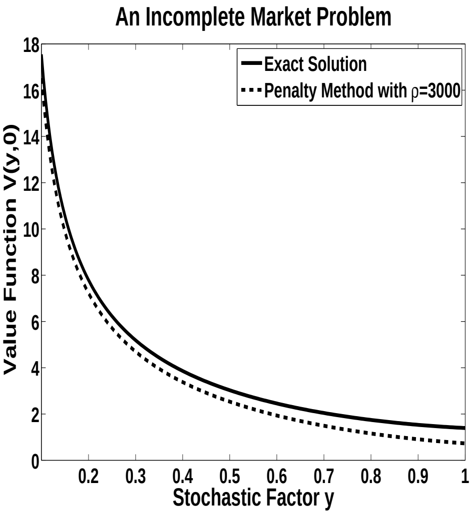
$$\begin{aligned} dS_t &= \mu S_t dt + \sigma(Y_t) S_t dW_t^1 \\ \text{and} \quad dY_t &= b(Y_t)dt + a(Y_t)dW_t^2, \end{aligned}$$

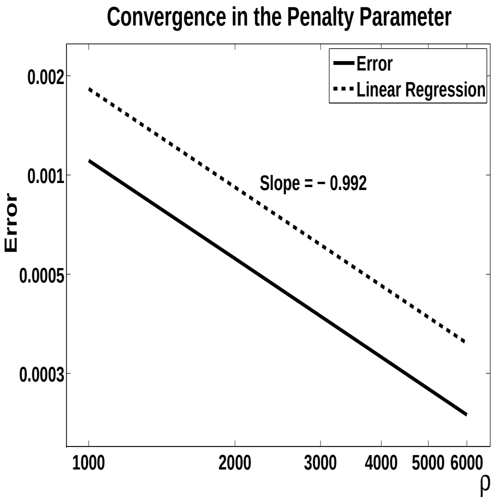
where $W_t^1$, $W_t^2$ are two Brownian motions correlated by $\varrho$.

The investor maximises terminal utility $U(x, y) = \frac{1}{\gamma} x^\gamma h(y)$, where $x$ is the current wealth.

The value function is (Zariphopoulou,'01) $v(x, y, t) = \frac{x^\gamma}{\gamma} V(y, t)$, where $V$ solves

$$\frac{1}{\gamma}\left[V_t + \frac{1}{2} a^2(y) V_{yy} + b(y) V_y\right] + rV$$
$$+ \max_{\pi \in \mathbb{R}} \left[\frac{1}{2}(\gamma - 1)\sigma^2(y)\pi^2 V + \varrho\sigma(y)a(y)\pi V_y + (\mu - r)\pi V\right] = 0.$$


An Incomplete Market Problem
Exact Solution
Penalty Method with ρ=3000
Value Function V(y,0)
Stochastic Factor y
0
2
4
6
8
10
12
14
16
18
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Convergence in the Penalty Parameter
Error
Linear Regression
Slope = − 0.992
Error
0.002
0.001
0.0005
0.0003
1000
2000
3000
4000
5000
6000
ρ

# Required Iterations

| *Policy Iteration* | $n = 1$ | $n = 2$ |
|---|---|---|
| $M, N = 50$ | 6% | 94% |
| $M, N = 200$ | 11% | 89% |
| $M = 200, N = 50$ | 53% | 47% |
| $M = 50, N = 200$ | - | 100% |

| *Penalty Method* $(\rho = 4e03)$ | $n = 1$ | $n = 2$ |
|---|---|---|
| $M, N = 50$ | 8% | 92% |
| $M, N = 200$ | 13% | 87% |
| $M = 200, N = 50$ | 49.5% | 50.5% |
| $M = 50, N = 200$ | - | 100% |

| *Penalty Method* $(\rho = 1e06)$ | $n = 1$ | $n = 2$ |
|---|---|---|
| $M, N = 50$ | 6% | 94% |
| $M, N = 200$ | 11% | 89% |
| $M = 200, N = 50$ | 55% | 45% |
| $M = 50, N = 200$ | - | 100% |

# Computational Time

| Grid Size | Policy | Penalty ($\rho = 4e03$) | Penalty ($\rho = 1e06$) |
|---|---|---|---|
| $M, N = 50$ | $1.08s$ | $1.08s$ | $1.09s$ |
| $M, N = 200$ | $35.07s$ | $34.76s$ | $35.16s$ |
| $M = 200,\ N = 50$ | $2.74s$ | $2.82s$ | $2.76s$ |
| $M = 50,\ N = 200$ | $10.89s$ | $10.88s$ | $10.91s$ |

# HJB Equations in Option Pricing

## Early Exercise in an Incomplete Market

Consider traded and non-traded asset $(S_t)$ and $(Y_t)$, respectively,

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sigma S_t\, dW_t^1 \\ \text{and} \quad dY_t &= b(Y_t)\, dt + a(Y_t)\, dW_t^2\,, \end{aligned}$$

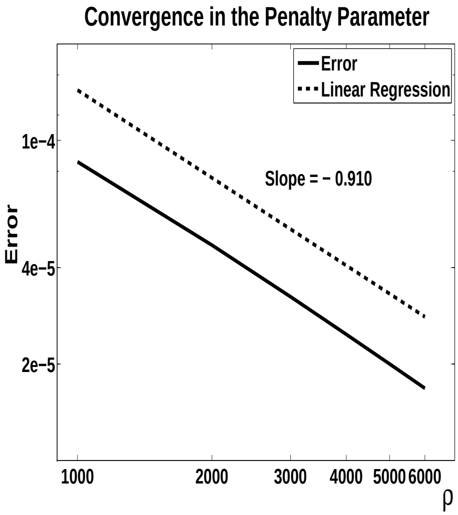
where $W_t^1$, $W_t^2$ are two Brownian motions correlated by $\varrho$, $r = 0$.

Assume the investor's utility function $U(x) = -e^{-\gamma x}$, $x$ wealth.

The buyer's early exercise indifference price of an American option with payoff $P$ on $(Y_t)$ solves (Oberman & Zariphopoulou,'03)

$$\min\Big\{ \max_{u \in \mathbb{R}} \mathcal{L}_u^b \psi \,, \psi - P(y) \Big\} = 0,$$

$$\begin{aligned} \text{where} \quad \mathcal{L}_u^b \psi := -\psi_t - \frac{1}{2} a^2(y)\psi_{yy} - \Big(b(y) - \rho\frac{\mu}{\sigma} a(y)\Big)\psi_y \\ + \frac{1}{2}\gamma(1-\rho^2)a^2(y)(2u\psi_y - u^2). \end{aligned}$$


Convergence in the Penalty Parameter
Error
Linear Regression
Slope = − 0.910
1e−4
4e−5
2e−5
Error
1000
2000
3000
4000
5000
6000
ρ

# Required Iterations

| *Policy Iteration* | Max Iterations | ∅ Iterations |
|---|---|---|
| $M, N = 50$ | 4 | 2.20 |
| $M, N = 200$ | 11 | 2.17 |
| $M = 200, N = 50$ | 4 | 1.83 |
| $M = 50, N = 200$ | 18 | 2.88 |
| *Penalty Method* ($\rho = 4e03$) | Max Iterations | ∅ Iterations |
| $M, N = 50$ | 3 | 1.98 |
| $M, N = 200$ | 3 | 1.21 |
| $M = 200, N = 50$ | 3 | 1.15 |
| $M = 50, N = 200$ | 4 | 2.16 |
| *Penalty Method* ($\rho = 1e06$) | Max Iterations | ∅ Iterations |
| $M, N = 50$ | 2 | 1.10 |
| $M, N = 200$ | 3 | 1.08 |
| $M = 200, N = 50$ | 2 | 1.02 |
| $M = 50, N = 200$ | 4 | 1.38 |

# Required Iterations

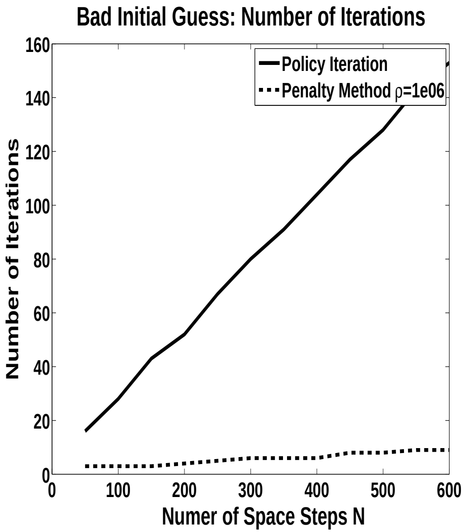
| *Policy Iteration* | Max Iterations | ∅ Iterations |
|---|---|---|
| $M = 1,\ N = 100$ | 28 | 28 |
| $M = 1,\ N = 300$ | 80 | 80 |

| *Penalty Method* $(\rho = 4e03)$ | Max Iterations | ∅ Iterations |
|---|---|---|
| $M = 1,\ N = 100$ | 4 | 4 |
| $M = 1,\ N = 300$ | 6 | 6 |

| *Penalty Method* $(\rho = 1e06)$ | Max Iterations | ∅ Iterations |
|---|---|---|
| $M = 1,\ N = 100$ | 3 | 3 |
| $M = 1,\ N = 300$ | 6 | 6 |

# Computational Time

| Grid Size | Policy | Penalty ($\rho = 4e03$) | Penalty ($\rho = 1e06$) |
|---|---|---|---|
| $M,\ N = 50$ | $0.38s$ | $0.25s$ | $0.17s$ |
| $M,\ N = 200$ | $6.42s$ | $4.09s$ | $3.82s$ |
| $M = 200,\ N = 50$ | $0.86s$ | $0.66s$ | $0.61s$ |
| $M = 50,\ N = 200$ | $2.12s$ | $1.47s$ | $1.13s$ |
| $M = 1,\ N = 100$ | $0.19s$ | $0.04s$ | $0.04s$ |
| $M = 1,\ N = 300$ | $2.18s$ | $0.19s$ | $0.19s$ |


Bad Initial Guess: Number of Iterations
Policy Iteration
Penalty Method ρ=1e06
Number of Iterations
Numer of Space Steps N
0
20
40
60
80
100
120
140
160
0
100
200
300
400
500
600

# Conclusion

- Monotone discretisation schemes converge to the financially relevant solutions to HJB equations. Penalisation is a powerful means of solving the resulting nonlinear discrete systems.

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**References:**


(1.) G. Barles, Convergence of numerical schemes for degenerate parabolic equations arising in finance, Numerical Methods in Finance, 1997.

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(3.) P. A. Forsyth, G. Labahn, Numerical methods for controlled HJB PDEs in finance, Journal of Computational Finance, 2007.

(4.) O. Bokanowski, S. Maroso and H. Zidani, Some convergence results for Howard's algorithm, SIAM Journal on Numerical Analysis, 2009.

(5.) J. H. Witte, C. Reisinger, A penalty method for the numerical solution of HJB equations in finance, SIAM Journal on Numerical Analysis, 2010.

(6.) J. H. Witte, C. Reisinger, Penalty Methods for the Numerical Solution of HJB Equations – Continuous Control and Obstacle Problems, Working Paper, 2011.

**Thank you.**

**References:**

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