

Lévy-Based Interest Rate Derivatives: Part I: Change of Time Method*

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Outline of Presentation

1. Literature Review and Bond Pricing
2. Stochastic Interest Rates (SIRs): Gaussian- and Lévy-Based
3. Change of Time Method and Pricing of Zero-Coupon Bonds
4. Conclusion

Literature Review

Introduction to Interest Rate Models: Rebonato (1996), Bingham and Kiesel (1998), Björk (1998), James and Webber (2000), Pelsser (2000), Brigo and Mercurio (2001), Filipovic (2001)

Two-Factor Interest Rate Models: Schwartz (1979), Courtadon (1982), Schaefer and Schwartz (1984)

Multifactor Affine-Yield Models: Dai and Singleton (2000)

One-Dimensional CIR Equation with time-varying parameters: Maghsoodi (1996)

Literature Review (cntd)

Common Short-Rate Models: Vasićek (1977), Cox-Ingersoll-Ross (1985), Ho and Lee (1986), Heath, Jarrow and Morton (1987), Black, Derman and Toy (1990), Black-Karasinski (1991), Longstaff and Schwartz (2001)

Empirical comparison of various Short Rate Models: Chan, Karoly, Longstaff and Sanders (1992)

Use of a Log-Normal Simple Interest Rate to Price Caps and Floors: Miltersen, Sandman and Sondermann (1997) LIBOR Term-Structure Model: Brace, Gatarek and Musiela (1997)

Swaps Interest Rate Market: Jamshidian (1990)

Literature Review (cntd)

Term-Structure Models with Jumps: Björk, Kabanov and Runggaldier (1997), Das (1999), Das and Foresi (1996), Glasserman and Kou (2003), Glasserman and Merener (2003), Shirakawa (1991)

Levy-Based Interest Rate Models: Eberlein and Raible (1999), Raible (2000)

Bond Pricing

A *bond* is a contract, paid for up-front, that yields a known amount on a known date in the future, $t = T$. If $P(t, T)$ is the value of zero-coupon bond at time t , then

$$P(t, T) = \exp\left(-\int_t^T r(s)ds\right),$$

where $r(t)$ is the instantaneous interest rate.

If there is no coupon the bond is known as a *zero-coupon bond*.

Bond Pricing (cntd)

Bond Pricing with Known Interest Rates:

Let $V(t, T)$ be the value of the bond contract, $t < T$. If the interest rate $r(t)$ and coupon payment $K(t)$ are known functions of time, the bond price is also a function of time only: $V = V(t, T)$. If this bond pays the owner Z at time $t = T$ then we know that $V(T, T) = Z$. We now derive an equation for the value of the bond at a time before maturity, $t < T$.

Suppose we hold one bond. Then arbitrage considerations lead us to the following equation:

$$\frac{dV}{dt} + K(t) = r(t)V.$$

($K(t)$ -coupon payment, $V(t, T)$ -value of bond contract)

Bond Pricing (cntd)

Yield Curve:

$$Y(t, T) = -\frac{\log(V(t, T)/V(T, T))}{T - t},$$

where t is the current time.

The Short Rate:

$$r(t) := Y(t, t) = \lim_{T \rightarrow t} Y(t, T)$$

Bond Pricing (cntd)

The *short rate* is the rate on instantaneous borrowing and lending. Historically, it was the short rate which was modelled as the basic process. In practice, this rate is stochastic and can fluctuate over time. Note that the short rate is actually a theoretical entity which does not exist in real life and can not be directly observed.

A sum of 1 invested in the short rate at time zero and continuously rolled over, i.e., instantaneously reinvested, is called the money-market account. Its value $S_0(t) = \exp[\int_0^t r(s)ds]$. If r is deterministic and constant, $S_0(t)$ reduces to the classical *bank account*: $S_0(t) = B(t) = \exp[rt]$.

Stochastic Interest Rates (SIRs)

Basic Model: Cox-Ingersoll-Ross (CIR) (1985). It is based on the CIR process:

$$dr(t) = k(\theta - r(t))dt + \gamma\sqrt{r}dW(t).$$

$P(t, u)$ -*price of zero-coupon bond*

$$P(t, u) = E^*[\exp(-\int_t^u r(s)ds)|\mathcal{F}_t]$$

Stochastic Interest Rates (SIRs)

"Riskless" asset

$$S_t^0 = \exp\left(\int_0^t r(s)ds\right)$$

Equation for $P(t, u)$

$$\frac{dP(t, u)}{P(t, u)} = (r(t) - \sigma(t, u)q(t))dt + \sigma(t, u)dW(t),$$

Risk-Neutral World:

$$\frac{dP(t, u)}{P(t, u)} = r(t)dt + \sigma(t, u)dW^*(t).$$

One-Factor and Multi-Factor Gaussian Interest Rate Models

One-Factor Gaussian SIRMs

1. *The Geometric Brownian Motion Model* (Rendleman and Bartter (1980)). $dr(t) = \mu r(t)dt + \sigma r(t)dW(t)$.
2. *The Ornstein-Uhlenbeck* (1930) Model. $dr(t) = -\mu r(t)dt + \sigma dW(t)$,
3. *The Vasiček* (1977) Model. $dr(t) = \mu(b - r(t))dt + \sigma dW(t)$.
4. *The Continuous-Time GARCH Model*. $dr(t) = \mu(b - r(t))dt + \sigma r(t)dW(t)$.

One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

One-Factor Gaussian SIRMs

5. *The Cox-Ingersoll-Ross* (1985) Model. $dr(t) = k(\theta - r(t))dt + \gamma\sqrt{r}dW(t)$.
6. *The Ho and Lee* (1986) Model. $dr(t) = \theta(t)dt + \sigma dW(t)$.
7. *The Hull and White* (1990) Model. $dr(t) = (a(t) - b(t)r(t))dt + \sigma(t)dW(t)$

One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

One-Factor Gaussian SIRMs

8. *The Heath, Jarrow and Morton* (1987) Model. Define the forward interest rate $f(t, s)$, for $t \leq s$, characterized by the following equality $P(t, u) = \exp[-\int_t^u f(t, s)ds]$ for any maturity u . $f(t, s)$ represents the instantaneous interest rate at time s as 'anticipated' by the market at time t . It is natural to set $f(t, t) = r(t)$. The process $f(t, u)_{0 \leq t \leq u}$ satisfies an equation $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dW(v)$, where the processes a and b are continuous. We note, that the last SDE may be written in the following form: $df(t, u) = b(f(t, u))(\int_t^u b(f(t, s)))ds + b(f(t, u))d\hat{W}(t)$, where $\hat{W}(t) = W(t) - \int_0^t q(s)ds$ and $q(t) = \int_t^u b(f(t, s))ds - \frac{a(t, u)}{b(f(t, u))}$.

One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Multi-Factor Gaussian SIRMs

Multi-factor models driven by Brownian motions can be obtained using various combinations of above-mentioned processes. We give one example of two-factor continuous-time GARCH SIRM:

$$\begin{cases} dr(t) &= \mu(b(t) - r(t))dt + \sigma r(t)dW^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW^2(t), \end{cases}$$

where W^1, W^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Change of Time Method for SDE driven by Brownian motion

Definition 1. A *time change* is a right-continuous increasing $[0, +\infty]$ -valued process $(T_t)_{t \in R_+}$ such that T_t is a stopping time for any $t \in R_+$. By $\hat{\mathcal{F}}_t := \mathcal{F}_{T_t}$ we define the time-changed filtration $(\hat{\mathcal{F}}_t)_{t \in R_+}$. The *inverse time change* $(\hat{T}_t)_{t \in R_+}$ is defined as $\hat{T}_t := \inf\{s \in R_+ : T_s > t\}$. (See Ikeda and Watanabe (1983)).

We consider the following SDE driven by a *Brownian motion*:

$$dX(t) = a(t, X(t))dW(t),$$

where $W(t)$ is a Brownian motion and $a(t, X)$ is a continuous and measurable by t and X function on $[0, +\infty) \times R$.

Change of Time Method for SDE driven by Brownian motion (cntd)

Theorem. (*Ikeda and Watanabe* (1981), Chapter IV, Theorem 4.3) Let $\hat{W}(t)$ be an one-dimensional \mathcal{F}_t -Wiener process with $\hat{W}(0) = 0$, given on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and let $X(0)$ be an \mathcal{F}_0 -adopted random variable.

Change of Time Method for SDE driven by Brownian motion (cntd)

Define a continuous process $V = V(t)$ by the equality

$$V(t) = X(0) + \widehat{W}(t).$$

Let T_t be the change of time process:

$$T_t = \int_0^t a^{-2}(T_s, X(0) + \widehat{W}(s)) ds.$$

If

$$X(t) := V(\widehat{T}_t) = X(0) + \widehat{W}(\widehat{T}_t)$$

and $\widehat{\mathcal{F}}_t := \mathcal{F}_{\widehat{T}_t}$, then there exists $\widehat{\mathcal{F}}_t$ -adopted Wiener process $W = W(t)$ such that $(X(t), W(t))$ is a solution of $dX(t) = a(t, X(t))dW(t)$ on probability space $(\Omega, \mathcal{F}, \widehat{\mathcal{F}}_t, P)$, where \widehat{T}_t is the inverse to T_t time change.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models

Solution of One-Factor Gaussian SIRMs Using CTM

We use change of time method (see Ikeda and Watanabe (1981)) to get the solutions to the following below equations (see Swishchuk (2007)). $W(t)$ below is a standard Brownian motion, and \hat{W} is a $(\hat{T}_t)_{t \in R_+}$ -adapted standard Brownian motion on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t \in R_+}, P)$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Solution of One-Factor Gaussian SIRMs Using CTM

1. *Geometric Brownian Motion.* $dr(t) = \mu r(t)dt + \sigma r(t)dW(t)$. Solution $r(t) = e^{\mu t}[r(0) + \hat{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t [r(0) + \hat{W}(\hat{T}_s)]^2 ds$.

2. *Ornstein-Uhlenbeck Process.* $dr(t) = -\mu r(t)dt + \sigma dW(t)$, Solution $r(t) = e^{-\mu t}[r(0) + \hat{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t (e^{\mu s}[r(0) + \hat{W}(\hat{T}_s)])^2 ds$.

3. *Vasićek Process.* $dr(t) = \mu(b - r(t))dt + \sigma dW(t)$, solution $r(t) = e^{-\mu t}[r(0) - b + \hat{W}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^2 \int_0^t (e^{\mu s}[r(0) - b + \hat{W}(\hat{T}_s)] + b)^2 ds$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models

Solution of One-Factor Gaussian SIRMs Using CTM (cntd)

4. *Continuous-Time GARCH Process.* $dr(t) = \mu(b - r(t))dt + \sigma r(t)dW(t)$. Solution $r(t) = e^{-\mu t}(r(0) - b + \widehat{W}(\widehat{T}_t)) + b$, where $\widehat{T}_t = \sigma^2 \int_0^t [r(0) - b + \widehat{W}(\widehat{T}_s) + e^{\mu s} b]^2 ds$.

5. *Cox-Ingersoll-Ross Process.* $dr^2(t) = k(\theta - r^2(t))dt + \gamma r(t)dW(t)$, solution $r^2(t) = e^{-kt}[r_0^2 - \theta^2 + \widehat{W}(\widehat{T}_t)] + \theta^2$, where $\widehat{T}_t = \gamma^{-2} \int_0^t [e^{kT_s}(r_0^2 - \theta^2 + \widehat{W}(s)) + \theta^2 e^{2kT_s}]^{-1} ds$.

6. *Ho and Lee Process.* $dr(t) = \theta(t)dt + \sigma dW(t)$. Solution $r(t) = r(0) + \widehat{W}(\sigma^2 t) + \int_0^t \theta(s)ds$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models

Solution of One-Factor Gaussian SIRMs Using CTM (cntd)

7. *Hull and White*. $dr(t) = (a(t) - b(t)r(t))dt + \sigma(t)dW(t)$.

Solution $r(t) = \exp[-\int_0^t b(s)ds][r(0) - \frac{a(s)}{b(s)} + \hat{W}(\hat{T}_t)]$,

where $\hat{T}_t = \int_0^t \sigma^2(s)[r(0) - \frac{a(s)}{b(s)} + \hat{W}(\hat{T}_s) + \exp[\int_0^s b(u)du] \frac{a(s)}{b(s)}]^2 ds$.

8. *Heath, Jarrow and Morton*. $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dW(v)$. Solution $f(t, u) = f(0, u) + \hat{W}(\hat{T}_t) + \int_0^t a(v, u)dv$, where $\hat{T}_t = \int_0^t b^2(f(0, u) + \hat{W}(\hat{T}_s) + \int_0^s a(v, u)dv)ds$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Solution of Multi-Factor Gaussian SIRMs Using CTM

Solution of *multi-factor models driven by Brownian motions* can be obtained using various combinations of solutions of the above-mentioned processes, see subsection 5.1, and CTM. We give one example of two-factor Continuous-Time GARCH model driven by Brownian motions:

$$\begin{cases} dr(t) &= \mu(b(t) - r(t))dt + \sigma r(t)dW^1(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW^2(t), \end{cases}$$

where W^1, W^2 may be correlated, $\mu, \xi \in \mathbb{R}, \sigma, \eta > 0$.

Solutions to the One-Factor and Multi-Factor Gaussian Interest Rate Models (cntd)

Solution of Multi-Factor Gaussian SIRMs Using CTM

Solution, using CTM for the first and the second equations, subsection 13.1:

$$r(t) = e^{-\mu t} [r(0) - e^{\xi t} (b(0) + \hat{W}^2(\hat{T}_t^2)) + \hat{W}^1(\hat{T}_t^1)] + e^{\xi t} [b(0) + \hat{W}^2(\hat{T}_t^2)],$$

where \hat{T}^i are defined in 4. ($i = 1$) and 1. ($i = 2$), respectively. Here, $W^1(t)$ and $W^2(t)$ are independent.

Lévy-based Stochastic Interest Rate Models (SIRMs)

Lévy Processes

Definition 2. By *Lévy process* we define a stochastically continuous process with *stationary and independent increments*, Sato (1999), Applebaum (2003), Schoutens (2003).

Examples of Lévy Processes:

- linear function $L(t) = bt$
- Brownian motion with drift
- Poisson process
- compound Poisson process

Lévy-based Stochastic Interest Rate Models (SIRMs)

Lévy-Khintchine Formula for Lévy Processes

$$E(e^{i(u, X(t))}) = \exp\left\{t\left[i(u, b) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} [e^{i(u, y)} - 1 - i(u, y)\mathbf{1}_{B_1(0)}] \nu(dy)\right]\right\}$$

(b, A, ν) -*Lévy-Khintchine triplet*

Lévy-based Stochastic Interest Rate Models (SIRMs)

Interpretation of Lévy-Khintchine triplet

- b stands for linear function, drift
- Diffusion matrix A stands for Brownian motion
- Lévy measure ν stands for jumps

Lévy-based Stochastic Interest Rate Models (SIRMs)

Lévy-Itô Decomposition

If X is a Lévy process, then there exists $b \in R^d$, a Brownian motion B_A with covariance matrix A and an independent Poisson random measure N on $R^+ \times (R^d - \{0\})$ such that, for each $t \geq 0$,

$$X(t) = bt + B_A(t) + \int_{|x|<1} x \tilde{N}(t, dx) + \int_{|x|\geq 1} x N(t, dx).$$

Lévy-based Stochastic Interest Rate Models (SIRMs)

Lévy Processes in Finance

- Brownian motion with drift (only continuous Lévy process)
- Merton model = Brownian motion + drift + Gaussian jumps
- Kou model = Brownian motion + drift + exponential jumps
- VG, IG, NIG, Generalized hyperbolic processes
- *α -stable Lévy processes*

Lévy-based Stochastic Interest Rate Models (SIRMs)

α -stable Lévy Processes

Definiton 3. Let $\alpha \in (0, 2]$. An *α -stable Lévy process* L such that L_1 (or equivalently any L_t) has a strictly α -stable distribution (i.e., $L_1 \equiv S_\alpha(\sigma, \beta, \nu)$) for some $\alpha \in (0, 2] \setminus \{1\}$, $\sigma \in R_+$, $\beta \in [-1, 1]$, $\nu = 0$ or $\alpha = 1$, $\sigma \in R_+$, $\beta = 0$, $\nu \in R$). (See Sato (1999)). We call L a *symmetric α -stable Lévy process* if the distribution of L_1 is even symmetric α -stable (i.e., $L_1 \equiv S_\alpha(\sigma, 0, 0)$ for some $\alpha \in (0, 2]$, $\sigma \in R_+$.) A process L is called $(T_t)_{t \in R_+}$ -adapted if L is constant on $[T_{t-}, T_t]$ for any $t \in R_+$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

α -stable Lévy Processes (cntd)

- the only self-similar Lévy processes
- either Brownian motion or pure jump
- characteristic exponent, Lévy-Khintchine triplet known in closed form
- 4 parameters
- infinite variance (except for Brownian motion)

Lévy-based Stochastic Interest Rate Models (SIRMs)

α -stable Lévy Processes (cntd)

- α -stable Lévy Processes are semimartingales ($\int_0^t f_s dL_s$ can be defined)
- α -stable Lévy Processes are pure discontinuous Markov processes with generator

$$Af(x) = \int_{\mathbb{R} - \{0\}} [f(x+y) - f(x) - yf'(y)\mathbf{1}_{|y|<1}(y)] \frac{K_\alpha}{|y|^{1+\alpha}} dy$$

- Lévy-Khintchine triplet is $(0, 0, \nu)$, where $\nu(dx) = \frac{\alpha dx}{\Gamma(1-\alpha)|x|^{1+\alpha}}$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy SIRMs

$L(t)$ below is a symmetric α -stable Lévy process. We define below various processes via SDE driven by α -stable Lévy process.

1. *Geometric α -stable Lévy Motion.* $dr(t) = \mu r(t-)dt + \sigma r(t-)dL(t).$
2. *Ornstein-Uhlenbeck Process Driven by α -stable Lévy Motion.*
 $dr(t) = -\mu r(t-)dt + \sigma dL(t).$

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy SIRMs

3. *Vasiček Process Driven by α -stable Lévy Motion.*

$$dr(t) = \mu(b - r(t-))dt + \sigma dL(t).$$

4. *Continuous-Time GARCH Process Driven by α -stable Lévy process.* $dr(t) = \mu(b - r(t-))dt + \sigma r(t-)dL(t).$

5. *Cox-Ingersoll-Ross Process Driven by α -stable Lévy Motion.*

$$dr(t) = k(\theta - r(t-))dt + \gamma\sqrt{r(t-)}dL(t).$$

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy SIRMs

6. *Ho and Lee Process Driven by α -stable Lévy Motion.*

$$dr(t) = \theta(t-)dt + \sigma dL(t).$$

7. *Hull and White Process Driven by α -stable Lévy Motion.*

$$dr(t) = (a(t-) - b(t-)r(t-))dt + \sigma(t)dL(t)$$

Lévy-based Stochastic Interest Rate Models (SIRMs)

One-Factor Lévy SIRMs

8. *Heath, Jarrow and Morton Process Driven by α -stable Lévy Motion.* Define the forward interest rate $f(t, s)$, for $t \leq s$, characterized by the following equality $P(t, u) = \exp[-\int_t^u f(t, s)ds]$ for any maturity u . $f(t, s)$ represents the instantaneous interest rate at time s as ‘anticipated’ by the market at time t . It is natural to set $f(t, t) = r(t)$. The process $f(t, u)_{0 \leq t \leq u}$ satisfies an equation $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dL(v)$, where the processes a and b are continuous.

Lévy-based Stochastic Interest Rate Models (SIRMs)

Multi-Factor Lévy SIRMs

Multi-factor models driven by α -stable Lévy motions can be obtained using various combinations of above-mentioned processes, see subsection 3.3. We give one example of two-factor continuous-time GARCH model driven by α -stable Lévy motions.

$$\begin{cases} dr(t) &= \mu(b(t-) - r(t-))dt + \sigma r(t-)dL^1(t) \\ db(t) &= \xi b(t-)dt + \eta b(t-)dL^2(t), \end{cases}$$

where L^1, L^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Lévy-based Stochastic Interest Rate Models (SIRMs)

Multi-Factor Lévy SIRMs

Also, we can consider various combinations of models, presented above, i.e., mixed models containing Brownian and Lévy motions. For example,

$$\begin{cases} dr(t) &= \mu(b(t-) - r(t-))dt + \sigma r(t-)dL(t) \\ db(t) &= \xi b(t)dt + \eta b(t)dW(t), \end{cases}$$

where Brownian motion $W(t)$ and Lévy process $L(t)$ may be correlated.

Change of Time Method for SDE Driven by Levy Motion

We denote by $L_{a.s.}^\alpha$ the family of all real measurable \mathcal{F}_t -adapted processes a on $\Omega \times [0, +\infty)$ such that for every $T > 0$, $\int_0^T |a(t, \omega)|^\alpha dt < +\infty$ a.s. We consider the following *SDE driven by a Lévy motion*:

$$dX(t) = a(t, X(t-))dL(t).$$

Change of Time Method for SDE Driven by Levy Motion

Theorem. (*Rosinski and Woyczynski (1986)*, Theorem 3.1., p.277). Let $a \in L_{a.s.}^\alpha$ be such that $T(u) := \int_0^u |a|^\alpha dt \rightarrow +\infty$ a.s. as $u \rightarrow +\infty$. If $\hat{T}(t) := \inf\{u : T(u) > t\}$ and $\hat{\mathcal{F}}_t = \mathcal{F}_{\hat{T}(t)}$, then the time-changed stochastic integral $\hat{L}(t) = \int_0^{\hat{T}(t)} a dL(t)$ is an $\hat{\mathcal{F}}_t - \alpha$ -stable Lévy process, where $L(t)$ is \mathcal{F}_t -adapted and $\mathcal{F}_t - \alpha$ -stable Lévy process. Consequently, a.s. for each $t > 0$ $\int_0^t a dL = \hat{L}(T(t))$, i.e., the stochastic integral with respect to a α -stable Lévy process is nothing but another α -stable Lévy process with randomly changed time scale.

Solutions of One-Factor Lévy-based SIRMs using CTM

Below we give the solutions to some one-factor Lévy IRM described by SDE driven by α -stable Lévy process. $L(t)$ below is a symmetric α -stable Lévy process, and \hat{L} is a $(\hat{T}_t)_{t \in \mathbb{R}_+}$ -adapted symmetric α -stable Lévy process on $(\Omega, \mathcal{F}, (\hat{\mathcal{F}}_t)_{t \in \mathbb{R}_+}, P)$.

1. *Geometric α -stable Lévy Motion.* $dr(t) = \mu r(t-)dt + \sigma r(t-)dL(t)$.

Solution $r(t) = e^{\mu t}[r(0) + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t [r(0) + \hat{L}(\hat{T}_s)]^\alpha ds$.

2. *Ornstein-Uhlenbeck Process Driven by α -stable Lévy Motion.*

$dr(t) = -\mu r(t-)dt + \sigma dL(t)$. *Solution* $r(t) = e^{-\mu t}[r(0) + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t (e^{\mu s}[r(0) + \hat{L}(\hat{T}_s)])^\alpha ds$.

Change of Time Method for SDE Driven by Levy Motion

3. *Vasiček Process Driven by α -stable Lévy Motion.* $dr(t) = \mu(b - r(t-))dt + \sigma dL(t)$. **Solution** $r(t) = e^{-\mu t}[r(0) - b + \hat{L}(\hat{T}_t)]$, where $\hat{T}_t = \sigma^\alpha \int_0^t (e^{\mu s}[r(0) - b + \hat{L}(\hat{T}_s)] + b)^\alpha ds$.

4. *Continuous-Time GARCH Process Driven by α -stable Lévy process.* $dr(t) = \mu(b - r(t-))dt + \sigma r(t-)dL(t)$. **Solution** $r(t) = e^{-\mu t}(r(0) - b + \hat{L}(\hat{T}_t)) + b$, where $\hat{T}_t = \sigma^\alpha \int_0^t [r(0) - b + \hat{L}(\hat{T}_s) + e^{\mu s}b]^\alpha ds$.

5. *Cox-Ingersoll-Ross Process Driven by α -stable Lévy Motion.* $dr(t) = k(\theta^2 - r(t-))dt + \gamma\sqrt{r(t-)}dL(t)$. **Solution** $r^2(t) = e^{-kt}[r_0^2 - \theta^2 + \hat{L}(\hat{T}_t)] + \theta^2$, where $\hat{T}_t = \gamma^{-\alpha} \int_0^t [e^{kT_s}(r_0^2 - \theta^2 + \hat{W}(s)) + \theta^2 e^{2kT_s}]^{-\alpha/2} ds$.

Change of Time Method for SDE Driven by Levy Motion

6. *Ho and Lee Process Driven by α -stable Lévy Motion.* $dr(t) = \theta(t-)dt + \sigma dL(t)$. **Solution** $r(t) = r(0) + \hat{L}(\sigma^2 t) + \int_0^t \theta(s)ds$.

7. *Hull and White Process Driven by α -stable Lévy Motion.* $dr(t) = (a(t-) - b(t-)r(t-))dt + \sigma(t-)dL(t)$.

Solution $r(t) = \exp[-\int_0^t b(s)ds][r(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T}_t)]$,

where $\hat{T}_t = \int_0^t \sigma^\alpha(s)[r(0) - \frac{a(s)}{b(s)} + \hat{L}(\hat{T}_s) + \exp[\int_0^s b(u)du] \frac{a(s)}{b(s)}]^\alpha ds$.

8. *Heath, Jarrow and Morton Process Driven by α -stable Lévy Motion.* $f(t, u) = f(0, u) + \int_0^t a(v, u)dv + \int_0^t b(f(v, u))dL(v)$. **Solution** $f(t, u) = f(0, u) + \hat{L}(\hat{T}_t) + \int_0^t a(v, u)dv$, where $\hat{T}_t = \int_0^t b^\alpha(f(0, u) + \hat{L}(\hat{T}_s) + \int_0^s a(v, u)dv)ds$.

Change of Time Method for SDE Driven by Levy Motion

Solution of Multi-Factor Lévy SIRMs Using CTM

Solution of multi-factor models driven by α -stable Lévy motions

can be obtained using various combinations of solutions of the above-mentioned processes and CTM. We give one example of two-factor continuous-time GARCH model driven by α -stable Lévy motions:

$$\begin{cases} dr(t) = \mu(b(t-) - r(t-))dt + \sigma r(t-)dL^1(t) \\ db(t) = \xi b(t-)dt + \eta b(t-)dL^2(t), \end{cases}$$

where L^1, L^2 may be correlated, $\mu, \xi \in R, \sigma, \eta > 0$.

Change of Time Method for SDE Driven by Levy Motion

Solution of Multi-Factor Lévy SIRMs Using CTM

Solution, using CTM for the first and the second equations:
$$r(t) = e^{-\mu t} [r(0) - e^{\xi t} (b(0) + \hat{L}^2(\hat{T}_t^2)) + \hat{L}^1(\hat{T}_t^1)] + e^{\xi t} [b(0) + \hat{L}^2(\hat{T}_t^2)],$$
where \hat{T}^i are defined in 4. ($i = 1$) and 1. ($i = 2$), respectively.

Bond Pricing

European Call Option

To calculate the value of a *European options* with maturity θ on the zero-coupon bond with maturity equal to T we could proceed as follows. If it is a call with strike price K , the value of the option at time θ is $\max(P(\theta, T) - K, 0)$ and it seems reasonable to hedge this call with a portfolio of riskless asset S_t^0 and zero-coupon bond $P(\theta, T)$ with maturity T . Using the classical arguments (and using the money-market account as numeraire), the price at time 0 of a European call option on a bond is given by

$$V(\theta, T, r) = E^* \left[\frac{1}{S_0(t)} \max(P(\theta, T) - K, 0) \right].$$

Bond Pricing (cntd)

Gaussian Bond Pricing for One-Factor SIRMs via CTM

The solution of the SDE

$$\frac{dP(t, u)}{P(t, u)} = r(t)dt + \sigma(t, u)dW^*(t).$$

can be written in the form

$$P(t, T) = P(0, T) \exp\left[\int_0^T r(s)ds\right] \frac{\exp\left[\int_0^t \sigma(s, T)dW^*(s)\right]}{E\left\{\exp\left[\int_0^t \sigma(s, T)dW^*(s)\right]\right\}}$$

$$= P(0, T) \exp\left[\int_0^T r(s)ds\right] \exp\left[\int_0^t \sigma(s, T)dW^*(s) - \frac{1}{2} \int_0^t \sigma(s, T)^2 ds\right].$$

We see that the log returns under risk-neutral measure approximately follow a Normally distributed random variable.

Bond Pricing (cntd)

Gaussian Bond Pricing for One-Factor SIRMs via PDE

Consider $V(t, T, r)$ -bond price at time t , where interest rate $r(t)$ follows the following SDE (in general form)

$$dr(t) = a(r, t)dt + b(r, t)dW(t).$$

For example, for GBM, $a = \mu r$ and $b = \sigma r$, for OU process, $a = -\mu r$ and $b = \sigma$, and so on.

Bond Pricing (cntd)

Gaussian Bond Pricing for One-Factor SIRMs via PDE

The zero-coupon bond pricing equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}b^2 \frac{\partial^2 V}{\partial r^2} + (a - \lambda b) \frac{\partial V}{\partial r} - rV = 0,$$

where the function λ is often called the market price of risk.

The final condition is

$$V(T, T, r) = Z.$$

This PDE may be solved approximately by standard numerical methods, see, for example, Wilmott, Howison and Dewynne (1995).

Remark. Empirical studies (see, for example, Raible (2000)) show that the normality assumption does not reflect reality. Empirically observed log returns of bonds turn out to have leptokurtic distribution.

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform

The zero-coupon bond price is modeled with the following process (see Eberlein and Raible (1999), Raible (2000)):

$$P(t, T) = P(0, T) \exp\left[\int_0^T r(s) ds\right] \frac{\exp\left[\int_0^t \sigma(s, T) dL(s)\right]}{E\left\{\exp\left[\int_0^t \sigma(s, T) dL(s)\right]\right\}},$$

where $r(t)$ has one of the representations mentioned above.

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

Eberlein and Raible (1999) derived the bond price process in the form

$$P(t, T) = P(0, T) \exp\left[\int_0^T r(s) ds\right] \frac{\exp\left[\int_0^t \sigma(s, T) dL(s)\right]}{\exp\left[\int_0^t \theta(\sigma(s, T)) dL(s)\right]},$$

where $\theta(u) := \log(E[\exp(uL(1))])$ denotes the logarithm of the moment-generating function of the Lévy process at time 1. For example, in the classical Gaussian model we choose $\theta(u) = u^2/2$ and $L(s) = W(s)$. We note, that we know the expressions for $r(t)$ in the above formula for many SIRMs.

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

Except when $L(t)$ is a Poisson or a Brownian motion, our *Lévy market model is an incomplete model*. It means that there are many different equivalent martingales measures to choose. In general this leads to many different possible prices for European options or bond options, etc.

One of the way to price bond is to use for the P^* the *Esscher transform* equivalent martingale measure. Following Gerber and Shiu (1994), we can by using the so-called Esscher transform find in some cases at least one equivalent martingale measure P^* .

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

Let $f(t, x)$ be the density of our model's (real world, i.e. under P) distribution of $L(t)$. For some real number $\theta \in \{\theta \in \mathbb{R} \mid \int_{-\infty}^{+\infty} \exp(\theta y) f(t, y) dy < +\infty\}$ we can define a new density

$$f^{(\theta)}(t, x) = \frac{\exp(\theta x) f(t, x)}{\int_{-\infty}^{+\infty} \exp(\theta y) f(t, y) dy}.$$

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

In order to assume finiteness of the expectation in the denominator above in the case of general Lévy processes, we assume that

$$\int_{\{|x|>1\}} \exp[vx] \nu(dx) < \infty, \quad \text{for } |v| < (1 + \epsilon)M,$$

where $\epsilon > 0$ and M is such that $0 \leq \sigma(s, T) \leq M$ (a.s.) for $0 \leq s \leq T$ and $\nu(dx)$ is the Lévy measure of L_1 (see Schoutens (2003)).

Lévy Bond Pricing for One-Factor SIRMs via CTM and Fourier Transform (cntd)

Another way to price bond is to consider *characteristic function (or Fourier transform)*, if it is known, of the risk-neutral log returns (see Carr and Madan (1998)). We note, that if we know the explicit expression for $r(t)$, then we can find the characteristic function of the risk-neutral bond price.

Conclusion

- Gaussian- and Levy-Based Interest Rate Models (IRMs)
- Change of Time Method for Gaussian- and Lévy-Based IRMs
- Pricing of Bond

The End

Thank You for Your Time and Attention!