Approximations of Security Markets by Geometric Markov Renewal Processes *

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Outline of Presentation

1. Intoduction

2. Random Evolutions (REe)

3. Applications of REs: Financial Mathematics, Biomathematics, etc.

4. Geometric Markov Renewal Processes (GMRP)

Outline of Presentation

5. Averaging of GMRP

6. Diffusion Approximation of GMRP

7. Normal Deviations of GMRP

8. Poisson Approximation of GMRP

9. Future Work

Random Evolutions

Random Evolutions (RE) are operator dynamical systems (in Banach or Hilbert spaces) where an operator depends on some stochastic process, Markov, semi-Markov, Lévy processes, etc. (see Korolyuk & Swishchuk (1992, 1995))

In physical language, a RE is a model for a dynamical system whose equation of state is subject to random variation.

In mathematical language, a RE is an operator differential equation with a generator depending on a parameter x, and this parameter is stochastic process.

Random Evolutions

The stochastic processes define the name for the REs: Markov, semi-Markov, Lévy, etc.

Also, depending on structure of operator equation, we have continuous REs, discontinuous (jump) REs, discrete RE, etc.

In this talk we shall mostly concentrate on semi-Markov jump REs.

Random Evolutions (RE)

RE = **Abstract Dynamical Systems** + **Random Media**

↓ ↓ ↓

dV(t,w)/dt=T(x(t,w))V(t,w) ← **Operator Evolution Equations dV(t)/dt= T(x)V(t)** + **Random Process**

T(x)V(t) ← **x(t,w)**

Applications of REs

Nonlinear Ordinary Differential Equations
dz/dt=F(z)

f(z(t))=V(t)f(z) →

Linear Operator Equation
df(z(t))/dt=F(z(t))df(z(t))/dz
dV(t)f/dt=TV(t)f
T:=F(z)d/dz

↓ **F=F(z,x)**
x=x(t,w)

Nonlinear Ordinary Stochastic Differential Equation
dz(t,w)/dt=F(z(t,w),x(t,w)))

f(z(t,w))=V(t,w)f(z) →

Linear Stochastic Operator Equation
dV(t,w)/dt=T(x(t,w))V(t.w)

Applications of REs: Traffic Process

$$\begin{cases} dz_t/dt &= v(z_t, x(t)) \\ z_0 = z \end{cases}$$

where

$$\Gamma(x)f(z) = v(z,x)df(z)/dz.$$

Applications of REs: Storage Process

$$z_t = z + \int_0^t v(z_s, x(s))ds + \sum_{k=1}^{\nu(t)} a(x_k),$$

where

$$\Gamma(x)f(z) = v(z,x)df(z)/dz \quad D(x,y)f(z) = f(z + a(x)).$$

Applications of REs: Risk Process

$$z_t = z + \int_0^t v(z_s, x(s))ds - \sum_{k=1}^{\nu(t)} a(x_k) = z + B(t) - A(t),$$

where

$$\Gamma(x)f(z) = v(z,x)df(z)/dz \quad D(x,y)f(z) = f(z + a(x)).$$

Applications of REs: Evolution of Biological Systems

$$y_{n+1} = y_n + g(y_n, x_n) - discrete \quad model$$

$$dy_t/dt = g(y_t, x(t)) - continuous - time \quad model$$

Applications of REs: Logistic Growth Model

$$dN_t/dt = r(x(t))N_t(1 - N_t/K).$$

Applications of REs: Financial Mathematics ((B,S,X)-Security Market or Geometric Regime-Switching Brownian Motion)

$$dB(t) = rB(t)dt, \quad B(0) > 0, r > 0, \quad -bond \quad price$$

$$dS(t) = \mu(x(t))S(t)dt + \sigma(x(t))S(t)dW(t) - stock \quad price$$

Random Evolutions: Another Names in the Literature

REs in Euclidean spaces have another names:

-hidden Markov (or other) models

-regime-switching models

Random Evolutions: Applications

REs have many applications:

-traffic theory
-storage teory
-risk theory
-biomathematics
-**financial mathematics**
-many others

In this talk we shall concentrate on financial applications of REs.

Geometric Markov Renewal Process (GMRP) as Jump Random Evolutions

We consider GMRP as an example of jump semi-Markov RE.

GMRP is a generalization of Aase (1988) geometric compound Poisson process in finance and Cox-Ross-Rubinstein (1976) discrete time model for the stock price.

Geometric Markov Renewal Process (GMRP)

Let $(x_k)_{k \in Z_+}$ be a Markov chain in phase space $(X, \mathcal{X})$ with transition probability $P(x, A)$, where $x \in X, A \in \mathcal{X}$. Let $(\tau_k)_{k \in Z_+}$ be a sequence of i.i.d. r.v. such that

$$P(\tau_{n+1} - \tau_k < t | x_n = x) = G_x(t), \tag{1}$$

where $x \in X, t \in R_+$.

Geometric Markov Renewal Process (GMRP)

Let us set

$$\theta_{n+1} := \tau_{n+1} - \tau_n,$$

$$\tau_n = \sum_{k=1}^{n} \theta_k,$$

and let

$$\nu(t) := \max\{n : \tau_n \leq t\} \tag{2}$$

be a *counting process.* The following process $(x_n; \theta_n)_{n \in Z_+}$ on phase states space $X \times R_+$ is called a *Markov renewal process* (MRP) [4]. The process $x(t) := x_{\nu(t)}$ is called a *semi-Markov process.*

Geometric Markov Renewal Process (GMRP)

Let $\rho(x)$ be a bounded continuous function on X such that $\rho(x) > -1$. The following functional on Markov renewal process $(x_n; \theta_n)_{n \in Z_+}$

$$S_t := S_0 \prod_{k=0}^{\nu(t)} (1 + \rho(x_k)), \tag{3}$$

where $S_0 > 0$, we call *geometric MRP* (GMRP).

We model stock price as S_t-GMRP.

GMRP as an Jump Random Evolutions

Let $C_0(R_+)$ be a space of continuous function on R_+, vanishing on the infinity, and let us define a family of contraction operators $D(x)$ on $C(R_+)$:

$$D(x)f(s) := f(s(1+\rho(x))).$$

Then, by definition, jump random evolution (RE) is defined as the following product:

$$V(t) = \prod_{k=1}^{\nu(t)} D(x_k).$$

GMRP as an Jump Random Evolutions

That is why, we obtain:

$$V(t)f(s) = \prod_{k=1}^{\nu(t)} D(x_k)f(s) = f(s\prod_{k=1}^{\nu(t)}(1+\rho(x_k)) = f(S(t)),$$

where $S(t)$ is defined in (3) and $S(0) = S_0 = s$. Let $Q(x, A, t)$ be a semi-Markov kernel for Markov renewal process $(x_n; \theta_n)_{n \in Z_+}$: $Q(x, A, t) = P(x, A)G_x(t)$.

GMRP as an Jump Random Evolutions

Let us define an expectation of jump RE $V(t)$:

$$u(t,x) := E_x[V(t)f(x(t))],$$

where $x(t) := x_{\nu(t)}$. Then function $u(t,x)$ satisfies the following Markov renewal equation (MRE):

$$u(t,x) - \int_0^t \int_X Q(x,dy,ds)D(y)u(t-s,y) = \bar{G}_x(t)f(x),$$

where $\bar{G}_x(t) = 1 - G_x(t)$, $\qquad G_x(t) := \mathcal{P}(\theta_{n+1} \leq t | X-n = x)$, $f(x)$ is a bounded and continuous function on X.

GMRP as an Jump Random Evolutions

Taking into account the above representation, we obtain that

$$w(t,x,s) := E_{x,s}[f(S(t),x(t))]$$

satisfies the following MRE:

$$w(t,x,s) - \int_0^t \int_X Q(x,dy,du)w(t-u,s(1+\rho(y)),y) = \widehat{G}_x(t)f(s,x),$$

where $f(s,x)$ is a bounded and continuous function on $R_+ \times X$.

GMRP as an Jump Random Evolutions

Equation (78) is a main tool in the investigation of limit distributions of the functional $S_T(t) = S_0 \prod_{k=1}^{\nu(tT)} (1 + \rho_T(x_k))$ as $T \rightarrow_{+\infty}$. It is one of the method for obtaining all the limits for $S_T(t)$ as $T \rightarrow +\infty$. The second method is martingale method.

Martingale Properties of GMRP: Infinitesimal Operator of the GMRP

Let us consider the representation

$$\ln \frac{S(t)}{S_0} = \sum_{k=1}^{\nu(t)} \ln(1 + \rho(x_k)).$$

To describe martingale properties of GMRP it needs to find an infinitesimal operator of the process

$$\eta(t) := \sum_{k=1}^{\nu(t)} \ln(1 + \rho(x_k)).$$

Martingale Properties of GMRP: Infinitesimal Operator of the GMRP

Let $\gamma(t) := t - \tau_{\nu(t)}$. Let us consider the process $(x(t), \gamma(t))$ on $X \times R_+$. It is Markov process with infinitesimal operator

$$\widehat{Q}f(x,t) := \frac{df}{dt} + \frac{g_x(t)}{\widehat{G}_x(t)} \int_X [P(x,dy)f(y,0) - f(x,t)],$$

where $g_x(t) := \frac{dG_x(t)}{dt}$, $\widehat{G}_x(t) = 1 - G_x(t)$, where $f(x,t) \in C(X \times R_+)$.

Martingale Properties of GMRP: Infinitesimal Operator of the GMRP

Infinitesimal operator for the process $\ln S(t)$ has the form:

$$\widehat{A}f(z,x) = \frac{g_x(t)}{\widehat{G}_x(t)}\int_X P(x,dy)[f(z+\ln(1+\rho(y),x)-f(z,x)],$$

where $z := \ln S_0$. The process $(\ln S(t), x(t), \gamma(t))$ is a Markov process on $R_+ \times X \times R_+$ with infinitesimal operator

$$\widehat{L}f(z,x,t) = \widehat{A}f(z,x,t) + \widehat{Q}f(z,x,t),$$

where operators $\widehat{A}$ and $\widehat{Q}$ are defined above.

Martingale Properties of GMRP: Infinitesimal Operator of the GMRP

From here we obtain that process

$$\widehat{m}(t) := f(\ln S(t), x(t), \gamma(t)) - f(z, x, 0) - \int_0^t (\hat{A} + \hat{Q}) f(\ln S(u), x(u), \gamma(u)) du$$

is an $\widehat{\mathcal{F}}_t$-martingale, where $\widehat{\mathcal{F}}_t := \sigma(x(s), \gamma(s); 0 \le s \le t)$. If $x(t) := x_{\nu(t)}$ is a Markov process with kernel

$$Q(x, A, t) = P(x, A)(1 - e^{-\lambda(x)t}),$$

namely, $G_x(t) = 1 - e^{-\lambda(x)t}$, then $g_x(t) = \lambda(x)e^{-\lambda(x)t}$, $\widehat{G}_x(t) = e^{-\lambda(x)t}$, and $\frac{g_x(t)}{\widehat{G}_x(t)} = \lambda(x)$.

Martingale Properties of GMRP: Infinitesimal Operator of the GMRP

That is why the operator $\widehat{A}$ has the form:

$$\widehat{A}f(z) = \lambda(x)\int_X P(x,dy)[f(z+\ln(1+\rho(y))) - f(z)].$$

For pair $(\ln S(t), x(t))$ on $R_+ \times X$ we obtain that this process is Markov process with infinitesimal operator

$$\widehat{L}f(z,x) = \widehat{A}f(z,x) + Qf(z,x),$$

where

$$Qf(z,x) = \lambda(x)\int_X P(x,dy)(f(y) - f(x)).$$

Martingale Properties of GMRP: Infinitesimal Operator of the GMRP

From here it follows that process

$$m(t) := f(\ln S(t), x(t)) - f(z, x) - \int_0^t (\widehat{A} + Q) f(\ln S(u), x(u)) du$$

is an $\mathcal{F}_t$-martingale, where $\mathcal{F}_t := \sigma(x(u); 0 \leq u \leq t)$.

Martingale property of the GMRP Let we have

$$S(t) = S_0 \prod_{k=1}^{\nu(t)} (1 + \rho(x_k)).$$

Let us define for all $t \in [0, T]$:

$$L_t := L_0 \prod_{k=1}^{\nu(t)} h(x_k), \qquad EL_0 = 1,$$

where $h(x)$ is a bounded continuous function:

$$\int_X h(y)P(x, dy) = 1, \qquad \int_X h(y)P(x, dy)\rho(y) = 0.$$

Martingale property of the GMRP

If $EL_T = 1$, then process $S(t)$ in (88) is an $(\mathcal{F}_t, P^*)$-martingale, where measure P^* is defined as follows

$$\frac{dP^*}{dP} = L_T,$$

and

$$\mathcal{F}_t := \sigma(x(s); 0 \leq s \leq t).$$

Martingale property of the GMRP

In discrete case we have

$$S_n = S_0 \prod_{k=1}^{\nu(t)} (1 + \rho(x_k)).$$

Let $L_n := L_0 \prod_{k=1}^{n} h(x_k)$, $\qquad EL_0 = 1$, where $h(x)$ is defined above. If $EL_N = 1$, then S_n is an $(\mathcal{F}_t, P^*)$-martingale, where $\frac{dP^*}{dP} = L_N$, and $\mathcal{F}_n := \sigma(x_k; 0 \le k \le n)$.

Geometric Compound Poisson Process as a GMRP

The GMRP process we call such by analogy with the geometric compound Poisson process

$$S_t = S_0 \prod_{k=1}^{N(t)} (1 + Y_k),$$

where $S_0 > 0$, $N(t)$ is a standard Poisson process, $(Y_k)_{k \in Z_+}$ are i.i.d. r.v., which is a trading model in many financial applications as a pure jump model (Aase, 1982).

The GMRP in (3) will be our main trading model in further analysis.

Cox-Ingersol-Ross Process for Stock Price as a GMRP

If S_n is the stock price at day n, then

$$S_n = S_0 \prod_{k=1}^{n} (1 + \rho_k),$$

where $\rho_k = a$ with probability $p > 0$, and $\rho_k = b$ with probability $1 - p$, where $-1 < a < r < b$ and $r > 0$ is the interest rate. This is Binomial model for stock price (Cox-Ross-Rubinstein, 1976).

Averaged GMRP

Further, we consider GMRP in series scheme. Under some conditions the averaged GMRPs as ergodic, merged and double-merged GMRP, are obtained.

Ergodic GMRP

Let $(x_n)_{n \in Z_+}$ has a stationary distribution $p(A), A \in \mathcal{X}$. The main proposition with respect to S_t consists of the fact that evolution of S_t takes place in stationary regime, when the effect of ergodicity of $(x_n)_{n \in Z_+}$ is sufficiently influenced. It means that S_t should be considered on enough large intervals of time. Obviously, that the same effect is reached under consideration $\nu(t)$ in another time scale , in "fast" time, we have enough large changes states of x_n.

Ergodic GMRP

Let us introduce the time scale interval $T > 0$ and consider $\nu_T(t) := \nu(tT)$, in new "fast" time. To avoid of infinite changes of S_t for finite time under increasing $T \to +\infty$, It is necessary to suppose dependence of value of jumps of the process S_t on T. It means that function $\rho(x)$ should depend on T (as $S_{\tau_k} - S_{\tau_k -} = S_{\tau_k -}\rho(x_k))$, i.e., $\rho \equiv \rho_T(x)$, in such way that $\rho_T(x) \to 0$ uniformly by x.

Ergodic GMRP

Namely, we suppose for simplicity that

$$\rho_T(x) = T^{-1}\rho(x),$$

for all $x \in X$. In this way, functional S_t in (3) has a form:

$$S_t^T = S_0 \prod_{k=0}^{\nu(tT)} (1 + \rho_T(x_k)) = S_0 \prod_{k=0}^{\nu(tT)} (1 + T^{-1}\rho(x_k)). \qquad (4)$$

Ergodic GMRP

Our aim is to study the behavior S_t^T in (4) in the following form:

$$S_t^T = S_0 \exp\{\sum_{k=0}^{\nu(tT)} \ln(1 + T^{-1}\rho(x_k))\}$$

or, equivalently,

$$\ln \frac{S_t^T}{S_0} = \sum_{k=0}^{\nu(tT)} \ln(1 + T^{-1}\rho(x_k)). \tag{5}$$

Ergodic GMRP

Ergodic GMRP has the following form:

$$\widehat{S}_t = S_0 e^{\widehat{\rho} t},$$

for all $t \in R_+$, and $S_0 > 0$, where $\widehat{\rho} := \int_X p(dx)\rho(x)/m$, $m(x) := \int_0^\infty (1 - G_x(t))dt$.

It means that the dynamic of ergodic GMRP which describes the dynamic of stock prices is the same as the dynamic of bond price with interest rate $\widehat{\rho}$.

Merged GMRP

Let us suppose that X consists of r ergodic classes X_i, $i = 1, 2, ..., r$, with stationary distributions $p_i(dx)$ in each class. Then Markov chain x_k is merged to the Markov chain $\widehat{x}_k$ in the merged phase space $\widehat{X} = \{1, 2, ..., r\}$. Taking into account the algorithms of phase merging (see Korolyuk & Limnios (2005)) we obtain that $T^{-1}\sum_{k=0}^{\nu(tT)} \rho(x_k)$ is merged to the integral functional

$$\tilde{\rho}(t) := \int_0^t \widehat{\rho}(\widehat{x}(s))ds, \tag{9}$$

where

$$\widehat{\rho}(k) := \int_{X_k} p_k(dx)\rho(x)/m(k), \tag{10}$$

and $m(k) := \int_{X_k} p_k(dx)m(x)$, $\widehat{x}(s)$ is a merged Markov process in phase states of space $\widehat{X}$.

Merged GMRP

In this way, we obtain from (5)-(7) that in merging scheme

$$\ln \frac{S_t^T}{S_0} \to_{T\to+\infty} \int_0^t \widehat{\rho}(\widehat{x}(s))ds.$$

From here we have that if $S_t^T \to_{T\to+\infty} \tilde{S}_t$, then

$$\tilde{S}_t = S_0 e^{\int_0^t \widehat{\rho}(\widehat{x}(s))ds}. \tag{11}$$

Remark. If $k = 1$ in (11), then $\tilde{S}_t = S_0 e^{\int_0^t \widehat{\rho}(\widehat{x}(s))ds} = S_0 e^{t\widehat{\rho}}$, where $\widehat{\rho}$ is defined in (10). Namely, as $k = 1$, then $\tilde{S}_t$ coincides with $\widehat{S}_t$ in (8).

Merged GMRP

Merged GMRP has the form:

$$\tilde{S}_t = S_0 e^{\int_0^t \widehat{\rho}(\widehat{x}(s))ds}, \tag{12}$$

where $t \in R_+$ and $S_0 > 0$. It means that the dynamic of merged GMRP is the same as the dynamic of bond price with various interest rates $\widehat{\rho}(k)$, where $k = 1, 2, ..., r$.

Double averaged GMRP Let us suppose that merged phase space $\widehat{X}$ of the merged Markov process $\widehat{x}(t)$ consists of one ergodic class with stationary distribution $(\widehat{p}_k)_{k=1,2,\ldots,N}$.

Double averaged GMRP

Then using algorithms of double averaging (see Korolyuk & Limnios (2005)) we will have for expression in (7) for large T :

$$T^{-1} \sum_{k=0}^{\nu(Tt)} \rho(x_k) \rightarrow_{T\rightarrow+\infty} t\check{\rho},$$

where $\check{\rho} := \sum_{i=1}^{r} \widehat{p}_k \widehat{\rho}(k)$, and $\widehat{\rho}(k)$ are defined in (10).

Double averaged GMRP

Then,

$$\ln \frac{S_t^T}{S_0} \to_{T\to+\infty} \ln \frac{\check{S}_t}{S_0} = t\check{\rho}.$$

Here, $\lim_{T\to+\infty} S_t^T := \check{S}_t$. That is why $\check{S}_t = S_0 e^{t\check{\rho}}$.

Double averaged GMRP

Double averaged GMRP has the form:

$$\check{S}_t = S_0 e^{t\check{\rho}},$$

where $t \in R_+$, and $S_0 > 0$. The dynamic of double averaged GMRP is the same as the dynamic of bond price with interest rate $\check{\rho}$.

Diffusion Approximation of GMRP

Under additional balance condition, averaging effect leads to diffusion approximation of GMRP. In fact, ν is considered in the new accelerated scale of time tT^2 : $\nu \equiv \nu(tT^2)$. Due to more rapid changings of states of the system under balance condition, the fluctuations are described by diffusion process.

Ergodic Diffusion Approximation

Let us suppose that balance condition is fulfilled for functional $S_t^T = S_0 \prod_{k=1}^{\nu(tT)} (1 + \rho_T(x_k))$:

$$\hat{\rho} = \int_X p(dx) \int_X P(x, dy)\rho(y)/m = 0. \qquad (17)$$

Then $\hat{S}_t = S_0$, for all $t \in R^+$.

Ergodic Diffusion Approximation

Consider S_t^T in the new scale of time tT^2 :

$$S_T(t) := S_{tT^2}^T = S_0 \prod_{k=1}^{\nu(tT^2)} (1 + T^{-1}\rho(x_k)). \tag{18}$$

Due to more rapid jumps of $\nu(tT^2)$ the process $S_T(t)$ will be fluctuated near the point S_0 as $T \to +\infty$.

Ergodic Diffusion Approximation

Using the reasonings analogic to (5)-(7), we obtain the following expression:

$$\ln \frac{S_T(t)}{S_0} = T^{-1} \sum_{k=1}^{\nu(tT^2)} \rho(x_k) - 1/2T^{-2} \sum_{k=1}^{\nu(tT^2)} \rho^2(x_k) + T^{-2} \sum_{k=1}^{\nu(tT^2)} r(T^{-1}\rho(x_k))\rho^2(x_k). \tag{19}$$

Ergodic Diffusion Approximation

Algorithms of ergodic averaging give the limit result for the second term in (19) (see Korolyuk & Swishchuk (1995)):

$$1/2T^{-2}\sum_{k=1}^{\nu(tT^2)}\rho^2(x_k)\to_{T\to+\infty}1/2t\widehat{\rho}^2, \tag{20}$$

where $\widehat{\rho}^2 := \int_X p(dx)\int_X P(x,dy)\rho^2(y)/m.$

Ergodic Diffusion Approximation

Using algorithms of diffusion approximation (see Korolyuk & Swishchuk (1995)) with respect to the first term in (19) we obtain:

$$T^{-1}\sum_{k=1}^{\nu(tT^2)} \rho(x_k) \to_{T\to+\infty} \sigma_\rho w(t), \tag{21}$$

where $\sigma_\rho^2 := \int_X p(dx)[1/2\int_X P(x,dy)\rho^2(y)$
$+\int_X P(x,dy)\rho(y)R_0P(x,dy)\rho(y)/m$, R_0 is a potential of $(x_n)_{n\in Z_+}$, $w(t)$ is a standard Wiener process.

Ergodic Diffusion Approximation

The last term in (19) goes to zero as $T \to +\infty$. Let $\widehat{S}(t)$ will be the limiting process for $S_T(t)$ in (19) as $T \to +\infty$.Then from (19)-(21) follows that

$$\ln \frac{S_T(t)}{S_0} \to_{T \to +\infty} \ln \frac{\widehat{S}(t)}{S_0} = \sigma_\rho w(t) - 1/2t\widehat{\rho}^2, \tag{22}$$

where σ_ρ and $\widehat{\rho}^2$ are defined in (21) and (20), respectively.

Ergodic Diffusion Approximation

In such a way, from (22) we obtain that

$$\widehat{S}(t) = S_0 e^{\sigma_\rho w(t) - 1/2 t \rho^2} = S_0 e^{-1/2 t \rho^2} e^{\sigma_\rho w(t)}.$$

That is why $\widehat{S}(t)$ satisfies the following stochastic differential equation (SDE):

$$d\widehat{S}(t) = \widehat{S}(t)[1/2(\sigma_\rho^2 - \widehat{\rho}^2)dt + \sigma_\rho dw(t)]. \tag{23}$$

Ergodic Diffusion Approximation

Ergodic diffusion GMRP has the form

$$\widehat{S}(t) = S_0 e^{-1/2t\widehat{\rho}^2} e^{\sigma_\rho w(t)}, \tag{24}$$

and satisfies the following SDE

$$\frac{d\widehat{S}(t)}{\widehat{S}(t)} = 1/2(\sigma_\rho - \widehat{\rho}^2)dt + \sigma_\rho dw(t).$$

European Call Option Pricing Formulas for Diffusion GMRP

As we have seen, ergodic diffusion GMRP $\widehat{S}(t)$ satisfies the following SDE:

$$\frac{d\widehat{S}(t)}{\widehat{S}(t)} = 1/2(\sigma_\rho - \widehat{\rho}_2)dt + \sigma_\rho dw(t),$$

where

$$\widehat{\rho}_2 = \int_X p(dx) \int_X P(x,dy)\rho^2(y)/m,$$

$$\sigma_\rho^2 = \int_X p(dx)[1/2 \int_X P(x,dy)\rho^2(y) + \int_X P(x,dy)\rho(y)R_0P(x,dy)\rho(y)/m.$$

European Call Option Pricing Formulas for Diffusion GMRP

The risk-neutral measure P^* for the process is:

$$\frac{dP^*}{P} = \exp\{-\theta t - \frac{1}{2}\theta^2 w(t)\},$$

where

$$\theta = \frac{\left(\frac{1}{2}(\sigma_\rho - \widehat{\rho}_2) - r\right)}{\sigma_\rho}.$$

Under P^*, the process $e^{-rt}\widehat{S}_t$ is a martingale and the process $w^*(t) = w(t) + \theta t$ is a Brownian motion. In this way, in the risk-neutral world, the process $\widehat{S}_t$ has the following form

$$\frac{d\widehat{S}(t)}{\widehat{S}(t)} = rdt + \sigma_\rho dw^*(t),$$

European Call Option Pricing Formulas for Diffusion GMRP

Using Black-Scholes formula we get the Europan call option pricing formula for our model:

$$C = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

where

$$d_+ = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma_\rho t)}{\sigma_\rho\sqrt{t}},$$

$$d_- = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma_\rho t)}{\sigma_\rho\sqrt{t}},$$

$\Phi(x)$ is a normal distribution and σ_ρ is defined above.

Merged Diffusion Approximation

Let us suppose that balance condition be satisfied:

$$\widehat{\rho}(k) = \int_{X_k} p_k(dx) \int_{X_k} P(x, dy)\rho(y)/m(k) = 0, \qquad (25)$$

for all $k = \overline{1, r}$, where $(x_k)_{k \in Z_+}$, p_k and $m(k)$ are defined in subsection 1.1.2 and conditions of reducibility of X are fulfilled.

Merged Diffusion Approximation

Merged diffusion approximated GMRP has the following form

$$\tilde{S}(t) = S_0 e^{-\frac{1}{2}\int_0^t \hat{\rho}^2(\hat{x}(s))ds + \int_0^t \hat{\sigma}_\rho(\hat{x}(s))dw(s)}. \tag{31}$$

and satisfies the stochastic differential equation (SDE):

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = \frac{1}{2}(\hat{\sigma}_\rho^2(\hat{x}(t)) - \hat{\rho}^2(\hat{x}(t)))dt + \hat{\sigma}_\rho(\hat{x}(t))dw(t), \tag{32}$$

where $\hat{x}(t)$ is a merged Markov process.

Diffusion Approximation under Double Averaging

Let us suppose that phase space $\widehat{X}$ of the merged Markov process $\widehat{x}(t)$ consists of one ergodic class with stationary distributions $(\widehat{p}_k; k = \overline{1, r})$. Let us also suppose that the balance condition is fulfilled:

$$\sum_{k=1}^{r} \widehat{p}_k \widehat{\rho}(k) = 0. \tag{33}$$

Diffusion Approximation under Double Averaging

Then the algorithms of DA under double averaging give the following result as $T \to +\infty$:

$$\ln \frac{S_T(t)}{S_0} \to_{T\to+\infty} \ln \frac{\check{S}(t)}{S_0} = \check{\sigma}_\rho w(t) - \frac{1}{2}\check{\rho}^2 t, \qquad (34)$$

where

$$\check{\sigma}_\rho^2 := \sum_{k=1}^{r} \widehat{p}_k \widehat{\sigma}_\rho^2(k), \qquad \check{\rho}^2 := \sum_{k=1}^{r} \widehat{p}_k \widehat{\rho}^2(k), \qquad (35)$$

and $\widehat{\rho}^2(k)$ and $\widehat{\sigma}_\rho^2(k)$ are defined in (27) and (29), respectively.

Diffusion Approximation under Double Averaging

Diffusion GMRP under double averaging has the form

$$\breve{S}(t) = S_0 e^{-\frac{1}{2}\breve{\rho}^2 t + \breve{\sigma}_\rho w(t)},$$

and satisfies the SDE

$$\frac{d\breve{S}(t)}{\breve{S}(t)} = \frac{1}{2}(\breve{\sigma}_\rho^2 - \breve{\rho}^2)dt + \breve{\sigma}_\rho dw(t).$$

Normal Deviations of GMRP

The above-considered algorithms of averaging define the averaged systems (or models) which may be considered as the first approximation.

The above-considered algorithms of DA under balance condition define diffusion models which may be considered as the second approximation.

In this section we consider the algorithms of construction of the first and the second approximation in the case when the balance condition is not fulfilled.

Ergodic Normal Deviations of GMRP Let us consider the *normal deviated process*

$$w_T(t) := \sqrt{T}(\alpha_T(t) - \widehat{\rho}t), \qquad (36)$$

where $\alpha_T(t) := T^{-1}\sum_{k=1}^{\nu(tT)} \rho(x_k)$, and $\widehat{\rho}$ is defined in (8) and $\widehat{\rho} \neq 0$. The process $w_T(t)$ defined deviations of the initial model $\alpha_T(t)$ in scale of time tT from the averaged model $\widehat{\rho}t$. It is known [2] that under large T the model $w_T(t)$ has the properties of Wiener process.

Ergodic Normal Deviations of GMRP

In this way, initial model under large T is represented in the form of double approximation:

$$\alpha_T(t) \simeq \widehat{\rho} t + \frac{1}{\sqrt{T}} \widehat{\sigma} \widehat{w}(t), \tag{37}$$

where $w(t)$ is a standard Wiener process with some diffusion coefficient $\widehat{\sigma}$.

Ergodic Normal Deviations of GMRP

Ergodic normal deviated GMRP has the form:

$$S_T(t) \simeq S_0 e^{\widehat{\rho} t + T^{-1/2} \widehat{\sigma} \widehat{w}(t)}, \tag{43}$$

or, in SDE form

$$\frac{dS_T(t)}{S_T(t)} = (\widehat{\rho} + \frac{1}{2} T^{-1} \widehat{\sigma}^2) dt + T^{-1/2} \widehat{\sigma} d\widehat{w}(t). \tag{44}$$

Reducible (or Merged) Normal Deviations

In reducible case we suppose that

$$\widehat{\rho}(k) \neq 0,$$

for all $k = \overline{1, r}$, and let us consider the normal deviated process

$$\tilde{w}_T(t) := \sqrt{T}(\alpha_T(t) - \tilde{\rho}(t)), \tag{45}$$

where $\alpha_T(t)$ is defined in (36), and $\tilde{\rho}(t) = \int_0^t \widehat{\rho}(\widehat{x}(s))ds$, and $\widehat{\rho}(k)$ is defined in (10).

Reducible (or Merged) Normal Deviated of GMRP

In this case the construction of normal deviated process for $\alpha_T(t)$ in reducible case consists of the fact (see Korolyuk & Swishchuk (1995)) that $\tilde{w}_T(t)$ is a stochastic Ito integral under large T :

$$\tilde{w}_T(t) \to_{T\to+\infty} \int_0^t \tilde{\sigma}(\hat{x}(s))dw(s), \tag{46}$$

where

$$\tilde{\sigma}^2(k) := \int_{X_k} p_k(dx)[(P\rho - \hat{\rho}(k))\mathbf{R}_o(P\rho - \hat{\rho}(k)) + 2^{-1}(P\rho - \hat{\rho}(k))^2]/m(k), \tag{47}$$

for all $k = \overline{1, r}$.

Reducible (or Merged) Normal Deviated of GMRP

Here, double approximation has the following form:

$$\alpha_T(t) \simeq \tilde{\rho}(t) + T^{-1/2} \int_0^t \tilde{\sigma}(\widehat{x}(s))dw(s). \tag{48}$$

Reducible (or Merged) Normal Deviated of GMRP

Reducible normal deviated GMRP has the form:

$$S_T(t) \simeq S_0 e^{\tilde{\rho}(t)+T^{-1/2}\int_0^t \tilde{\sigma}(\widehat{x}(s))dw(s)},$$

or, in the form of SDE

$$\frac{dS_T(t)}{S_T(t)} \simeq (\widehat{\rho}(\widehat{x}(t)) + \frac{1}{2}T^{-1}\tilde{\sigma}^2(\widehat{x}(t)))dt + T^{-1/2}\tilde{\sigma}(\widehat{x}(t))dw(t), \quad (49)$$

where $\widehat{\rho}(k)$ and $\tilde{\sigma}^2(k)$ are defined in (10) and (47), respectively.

Normal Deviations of GMRP under Double Averaging

From double averaging it follows that limiting process under double averaging is the process $\breve{\rho}t$, where $\breve{\rho}$ is defined in (13). Let us suppose that $\breve{\rho} \neq 0$, and let us consider the normal deviated process

$$\breve{w}_T(t) := \sqrt{T}(\alpha_T(t) - \breve{\rho}t). \tag{50}$$

Normal Deviations of GMRP under Double Averaging

Normal deviations of the initial process under double averaging consists of the fact that process $\breve{w}_T(t)$ in (50) under large T is a Wiener process with diffusion coefficient $\breve{\sigma}$:

$$\breve{\sigma}^2 := \sum_{k=1}^{r} \widehat{p}_k \tilde{\rho}^2(k), \tag{51}$$

where $\tilde{\sigma}^2(k)$ is defined in (47), namely,

$$\breve{w}_T(t) \to_{T\to+\infty} \breve{\sigma} w(t).$$

Normal Deviations of GMRP under Double Averaging

In this way, double approximation of $\alpha_T(t)$ in (50) is expressed in the form:

$$\alpha_T(t) \simeq \breve{\rho}t + T^{-1/2}\breve{\sigma}w(t). \tag{52}$$

From (7), (50) and (52) it follows that

$$\ln \frac{S_T(t)}{S_0} \simeq \breve{\rho}t + T^{-1/2}\breve{\sigma}w(t),$$

or, equivalently,

$$S_T(t) \simeq S_0 e^{\breve{\rho}t + T^{-1/2}\breve{\rho}w(t)}.$$

Normal Deviations of GMRP under Double Averaging

Normal deviated GMRP under double averaging has the form:

$$S_T(t) \simeq S_0 e^{\breve{\rho} t + T^{-1/2} \breve{\sigma} w(t)}, \tag{53}$$

or, in the form of SDE,

$$\frac{dS_T(t)}{S_T(t)} = (\breve{\rho} + \frac{1}{2} T^{-1} \breve{\sigma}^2) dt + T^{-1/2} \breve{\sigma} dw(t). \tag{54}$$

Averaging in Poisson Scheme

In this section we consider averaging of GMRP in Poisson scheme. In the limit we obtain compound Poisson process with deterministic drift.

Let $\rho_k^T(x,\omega) \equiv \rho_k^T(x)$ be a sequence of random variables for all $x \in X$ and for all $T > 0$. Let us consider the process S_t^T in series scheme:

$$S_t^T = S_0 \prod_{k=1}^{\nu(tT)} (1 + \rho_k^T(x_k,\omega)) = S_0 \prod_{k=1}^{\nu(tT)} (1 + T^{-1}\rho_k(x_k)). \qquad (55)$$

Averaging in Poisson Scheme

We note that this scheme is more general then previous, becouse of the random variables $\rho_k^T(x,\omega)$. From (55) it follows that

$$\ln \frac{S_t^T}{S_0} = \sum_{k=1}^{\nu(tT)} \ln(1+\rho_k^T(x_k)) \equiv T^{-1} \sum_{k=1}^{\nu(tT)} \rho_k(x_k), \qquad (56)$$

for large T.

Averaging in Poisson Scheme

Using the result by Korolyuk & Limnios (2002) we obtain that right hand-side of (56) converges weakly to compound Poisson process $P(t)$ with deterministic drift:

$$\sum_{k=1}^{\nu(tT)} \rho_k^T(x_k) \to_{T \to +\infty} P(t) := \sum_{k=1}^{N_0(t)} \alpha_k^0 + a_0 q t, \qquad (59)$$

where α_k^0 are i.i.d.r.v. with distribution function $F^0(z)$.

Averaging in Poisson Scheme

In this way,

$$\ln \frac{S_t^T}{S_0} \rightarrow_{T\rightarrow+\infty} \ln \frac{S_P(t)}{S_0} = P(t) = \sum_{k=1}^{N_0(t)} \alpha_k^0 + a_0 q t,$$

or,

$$S_P(t) = S_0 e^{\sum_{k=1}^{N_0(t)} \alpha_k^0 + a_0 q t},$$

where we indicate by $S_P(t)$ the limit $\lim_{T\rightarrow+\infty} S_t^T := S_P(t)$.

Averaging in Poisson Scheme

We note that

$$\sum_{k=1}^{N_0(t)} \alpha_k^0 = \int_0^t \int_0^{+\infty} y\mu(dy; ds),$$

where μ is a measure of jumps of the process $N_0(t)$. Hence,

$$\ln \frac{S_P(t)}{S_0} = \int_0^t \int_0^{+\infty} y\mu(dy; ds) + a_0 qt,$$

or, in the from of SDE

$$\frac{dS_P(t)}{S_P(t)} = \int_0^{+\infty} y\mu(dy; dt) + a_0 q dt. \tag{61}$$

Poisson GMRP is the solution of the SDE in (61). It means that the dynamic of stock price in Poisson scheme is described by Poisson GMRP and is the Merton model.

Proving the Results

All the above-mentioned results are obtained from the general results for semi-Markov random evolutions (see Korolyuk & Swishchuk (1995)) in series scheme:

- Weak convergence of S_t^T in Skorokhod space $D_R[0,+\infty)$
- Solution of martingale problem for the limit process $\widehat{S}_t$
- Uniqueness of solution of martingale problem

Option pricing formula under Poisson scheme

In this section we consider oprion pricing formula for European call options, which describe by dynamic of stock prices as GMRP in discrete and continuous time cases. Let $f(S_N) = (S_N - K)^*$, and $S_N = S_0 \prod_{k=1}^{N}(1 + \rho(x_k))$. Then

$$\begin{aligned} C_N(y) &= E^*[(1+r)^{-N} f(S_N)|\mathcal{F}_0] \\ &= (1+r)^{-N} \int_X \dots \int_X f(S_0 \prod_{i=1}^{N}(1+\rho(y_i))) P^*(y_{i-1}, dy_i), \end{aligned}$$

where $y_0 = y$, and $P^*(x, A) = hP$ ia the distribution of x_n with respect to P^*.

Option pricing formula under Poisson scheme

Let $f(S_T) = (S_T - K)^+$, and $S_T = S_0 \prod_{k=1}^{\nu(tT)} (1 + \rho(x_k))$. Then

$$\begin{aligned} C_T(y) &= E^*[(1+r)^{-N} f(S_T)|\mathcal{F}_0] \\ &= E^*[E^*[(1+r)^{-N} f(S_T)|\nu(T)|\mathcal{F}_0]] = \\ &= (1+r)^{-N} \sum_{k=0}^{+\infty} \mathcal{P}(\nu(T) = k) \\ & \quad \int_X \dots \int_X f(S_0 \prod_{i=1}^{k} (1 + \rho(y_i)) P^*(y_{i-1}, dy_i)), \end{aligned}$$

where $P^*(x, A) = hP$ is a distribution of x_n with respect to P^*, h is defined above, $y_0 = y$.

Option pricing formula under Poisson scheme

In the case of Poisson process $\nu(t) \equiv N(t)$ we obtain:

$$\begin{aligned} C_T(y) &= (1+r)^{-N} \sum_{k=0}^{+\infty} \frac{e^{-\lambda T}(\lambda T)^k}{k!} \int_X \\ \ldots & \int_X f(S_0 \prod_{i=1}^{k}(1+\rho(y_i))) P^*(y_{i-1}, dy_i). \end{aligned}$$

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Conclusion

1. Random Evolutions (REe)

2. Applications of REs: Financial Mathematics, Biomathematics, etc.

3. Geometric Markov Renewal Processes (GMRP)

4. Averaging, DA, ND and Poisson Approximation of GMRP

The End!

Thank you for your attention and time!

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