Portfolio optimization under downside risk measures

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Overview

This talk will contain an overview of the programme of research reported in the following papers:

D-V/W (2006) Asymptotic behaviour of mean-quantile efficient portfolios, *Finance and Stochastics* 10:529–551

D-V/W/Lari-Lavassani/Li (2010) Continuous-time portfolio selection under Conditional Capital at Risk, *Journal of Probability and Statistics* (to appear)

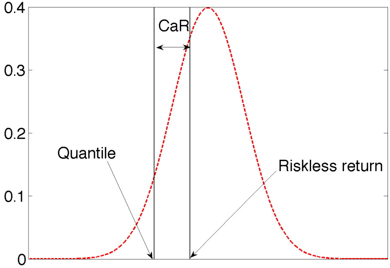
D-V/W (2010) Optimal portfolios of mean-reverting instruments, *Submitted*

Definitions

Capital at Risk (CaR)

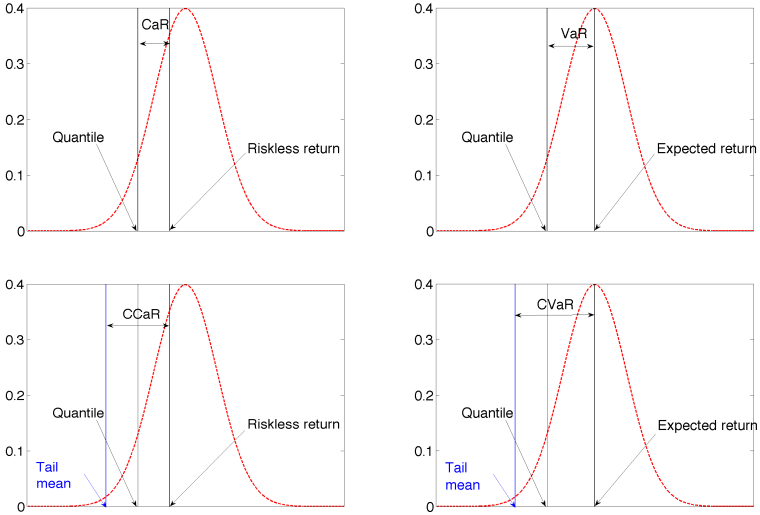
For a given α, CaR is the difference between the riskless return and q_α, i.e.

$$\mathrm{CaR}(V(t)) := V(0)R_0(t) - q_\alpha.$$



Value at Risk (VaR)

Comparison



0.4
0.3
0.2
0.1
0
CaR
Quantile
Riskless return
VaR
Expected return
CCaR
Tail
mean
CVaR

Coherency

Let $\mathcal{V}$ be the set of real valued random processes on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $S(t), Y(t) \in \mathcal{V}$, and $C \in \mathbb{R}$. Then $\rho : \mathcal{V} \to \mathbb{R}$ is a coherent risk measure if it has the following properties:

Subadditivity: $\rho(S(t) + Y(t)) \leq \rho(S(t)) + \rho(Y(t))$ for $t \in [0, T]$.

Positive Homogeneity: if $C > 0$, $\rho(CS(t)) = C\rho(S(t))$ for $t \in [0, T]$.

Monotonicity: if $S(t)) \geq Y(t)$ a.e., $\rho(S(t)) \leq \rho(Y(t))$ for $t \in [0, T]$.

Translation Invariance: $\rho(S(t) + C) = \rho(S(t))$ for $t \in [0, T]$.

	CaR	CCaR	VaR	CVaR
Subadditivity	no	yes	no	yes
Positive Homogeneity	yes	yes	yes	yes
Monotonicity	yes	yes	no	no
Translation Invariance	yes	yes	yes	yes

Coherency

Example

Let A and B be discrete random variables with joint probability distribution

$B \backslash A$	70	90	100
70	0	0	0.03
90	0	0	0.02
100	0.03	0.02	0.90

Then

$$A + B : \begin{pmatrix} 170 & 190 & 200 \\ 0.06 & 0.04 & 0.9 \end{pmatrix}$$

$$q_A = q_B = 90$$

$$q_{A+B} = 170 < q_A + q_B$$

$$\text{VaR}(A) = \text{VaR}(B) = 8.9.$$

$$\text{VaR}(A+B) = E[A+B] - q_{(A+B)} = 27.8 > \text{VaR}(A) + \text{VaR}(B)$$

Market setting

Bond Dynamics

$$dS_0(t) = r(t)S_0(t)dt, \qquad r(t) > 0, \quad t \in [0,T], \qquad S_0(0) = p_0 > 0.$$

Stock Dynamics

$$dS_i(t) = S_i(t)\left(b_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)\right), \qquad S_i(0) > 0.$$

Parameters

- $\sigma(t)$, $\sigma^{-1}(t)$, $b(t)$ and $r(t)$ are deterministic, Borel measurable, bounded functions.
- For each t, $\sigma(t)$ satisfies the non-degeneracy condition

$$\exists \delta > 0 : x'\sigma(t)\sigma(t)'x \geq \delta x'x, \quad , \forall x \in \mathbb{R}^m.$$

Portfolio dynamics

Trading strategy: $N(t) := (N_0(t), ..., N_m(t))'$.

Wealth process: $X(t) := \sum_{i=0}^{m} N_i(t) S_i(t)$.

Self-financing strategy: $dX(t) := \sum_{i=0}^{m} N_i(t) dS_i(t)$.

Risk premium: $B(t) = b(t) - r(t)\mathbf{1}$.

Portfolio process: $\pi_i(t) = \frac{N_i(t)S_i(t)}{X(t)}, \quad i = 1, \ldots, N$.

Wealth Equation

$$dX^\pi(t) = X^\pi(t)\Big(\big(r(t) + B(t)'\pi(t)\big)\, dt + \sigma(t)'\pi(t) dW(t)\Big),$$

$$X^\pi(0) = X_0.$$

Market Price of Risk

$$\theta(t) = \sigma(t)^{-1} B(t).$$

Quantile-based risk measures

$$\mathbb{E}\left[X^{\pi}(t)\right] = X_0 \exp\left(R_0(t) + \langle B, \pi\rangle_t\right).$$

α-quantile

$$q_\alpha(X_0, \pi, t) = X_0 \exp\left(R_0(t) + \langle B, \pi\rangle_t\right) \exp\left(-\tfrac{1}{2}\|\pi'\sigma\|_t^2 - |z_\alpha|\|\pi'\sigma\|_t\right).$$

Capital at Risk

$$\mathrm{CaR}(X_0, \pi, t) = X_0 \exp\left(R_0(t)\right)\left(1 - \exp(\langle B, \pi\rangle_t - \tfrac{1}{2}\|\sigma'\pi\|_t^2 - |z_\alpha|\|\sigma'\pi\|_t)\right).$$

Value at Risk

$$\mathrm{VaR}(X_0, \pi, t) = X_0 \exp\left(R_0(t) + \langle B, \pi\rangle_t\right)\left(1 - \exp(-\tfrac{1}{2}\|\sigma'\pi\|_t^2 - |z_\alpha|\|\sigma'\pi\|_t)\right).$$

For a vector-valued functions f and g on $[0, T]$, $\|f\|_t := \sqrt{\int_0^t \|f(s)\|^2\, ds}$, where $\|\cdot\|$ is the usual Euclidean norm. $\langle f, g\rangle_t$ is the corresponding inner product.

Quasiconvexity

Let S be a nonempty convex set in R^m. The function $f : S \to R$ is said to be strongly quasiconvex if, for each $x_1, x_2 \in S$, $x_1 \neq x_2$, the inequality

$$f(\lambda x_1 + (1-\lambda)x_2) < \max\{f(x_1), f(x_2)\},$$

holds for each $\lambda \in (0, 1)$.

Theorem

CaR is a strongly quasiconvex function of $\pi(\cdot)$, i.e.

$$CaR(X_0, \lambda\pi + (1-\lambda)\xi, T) < \max\{CaR(X_0, \pi, T), CaR(X_0, \xi, T)\},$$

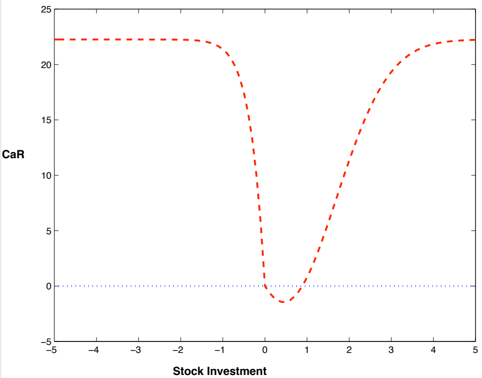
for all $\pi(t)$, $\xi(t)$ such that $\{t \in [0, T] \mid \pi(t) \neq \xi(t)\}$ has a positive Lebesgue measure, and for all $\lambda \in (0, 1)$.

Quasiconvexity

Example (Quasiconvexity of CaR)

We consider a market with one stock following the SDE

$$dS(t)=S(t)\left(0.15dt+0.2dW(t)\right),\ t\in[0,16],\ S(0)=10.$$

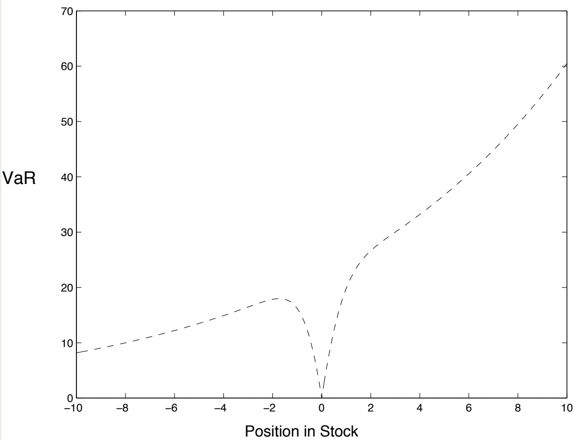


The graph of $\mathrm{CaR}(\pi)$ for $\pi\in[-5,5]$, $T=16$, $S(0)=10$, $r=0.05$, $\sigma=0.2$, $b=0.15$. The global minimum π_ε lies in the interval $[0,1]$, and satisfies $\mathrm{CaR}\left(\pi_\varepsilon\right)<0$.

VaR is not a strongly quasiconvex function of $\pi(\cdot)$

Example

$dS(t) = S(t)\,(0.15dt + 0.2dW(t))$, $t \in [0, 16]$, $S(0) = 10$, $r = 0.05$.



The Merton (or market) portfolio

Every optimal portfolio will involve a multiple of this.

$$\pi_M(t) = \left(\sigma(t)\sigma(t)'\right)^{-1} B(t).$$

$$\langle B, \pi_M \rangle_t = \|\theta\|_t^2 .$$

Minimal CaR

The portfolio that minimizes CaR (for a given α) is

$$\pi_{\epsilon_1} = \frac{\epsilon_1}{\|\theta\|_T} \pi_M(t),$$

where

$$\epsilon_1 = \left(\|\theta\|_T - |z_\alpha| \right)_+ .$$

The minimal CaR is

$$\mathrm{CaR}\big(X^{\pi_{\epsilon_1}}(T)\big) = X_0 R_0(T) \big(1 - \mathrm{e}^{\epsilon_1^2/2}\big),$$

and the corresponding expected wealth is

$$\mathbb{E}\big[X^{\pi_{\epsilon_1}}(T)\big] = X_0 R_0(T) \mathrm{e}^{\epsilon_1 \|\theta\|_T}.$$

Maximum expected wealth under constrained CaR

Suppose that C satisfies

$$\left(1 - \mathrm{e}^{\frac{1}{2}(\|\theta\|_T - |z_\alpha|)^2}\right)_+ \leq \frac{C}{X_0 R_0(T)} < 1.$$

The portfolio that maximizes $\mathbb{E}[X^\pi(T)]$ under the constraint $\mathrm{CaR}\big(X^\pi(T)\big) \leq C$ is

$$\pi_{\epsilon_2} = \frac{\epsilon_2}{\|\theta\|_T} \pi_M(t),$$

where

$$\epsilon_2 = \|\theta\|_T - |z_\alpha| + \sqrt{(\|\theta\|_T - |z_\alpha|)^2 - 2c},$$

where

$$c = \ln\left(1 - \frac{C}{X_0 R_0(T)}\right).$$

The corresponding expected wealth is

$$\mathbb{E}\left[X^{\pi_{\epsilon_2}}(T)\right] = X_0 R_0(T) \mathrm{e}^{\epsilon_2 \|\theta\|_T}.$$

Maximum expected wealth under constrained VaR

The portfolio that maximizes $\mathbb{E}[X^{\pi}(T)]$ under the constraint $\mathrm{VaR}\big(X^{\pi}(T)\big) \leq C$ is

$$\pi_{\epsilon_3} = \frac{\epsilon_3}{\|\theta\|_T}\pi_M(t),$$

where ϵ_3 solves

$$\mathrm{e}^{\epsilon_3\|\theta\|_T}\left(1 - \mathrm{e}^{-\epsilon_3^2/2-|z_\alpha|\epsilon_3}\right) = \frac{C}{X_0 R_0(T)}.$$

The corresponding expected wealth is

$$\mathbb{E}\big[X^{\pi_{\epsilon_3}}(T)\big] = X_0 R_0(T)\mathrm{e}^{\epsilon_3\|\theta\|_T}.$$

Minimal CCaR

The portfolio that minimizes CCaR (for a given α) is

$$\pi_{\epsilon_4} = \frac{\epsilon_4}{\|\theta\|_T} \pi_M(t),$$

where, if $\alpha \|\theta\|_T > \phi(|z_\alpha|)$, ϵ_4 solves

$$\phi(|z_\alpha| + \epsilon_4) = \Phi(-|z_\alpha| - \epsilon_4) \|\theta\|_T .$$

The minimal CCaR is

$$\mathrm{CCaR}\big(X^{\pi_{\epsilon_1}}(T)\big) = X_0 R_0(T) \left(1 - \frac{1}{\alpha} \mathrm{e}^{\epsilon_4 \|\theta\|_T} \Phi(-|z_\alpha|)\right),$$

and the corresponding expected wealth is

$$\mathbb{E}\big[X^{\pi_{\epsilon_4}}(T)\big] = X_0 R_0(T) \mathrm{e}^{\epsilon_4 \|\theta\|_T}.$$

Maximum expected wealth under constrained CCaR

Suppose that C satisfies

$$\begin{cases} X_0R(T)\left(1-\frac{1}{\alpha}\mathrm{e}^{\epsilon_4\|\theta\|_T}\Phi(-|z_\alpha|-\epsilon_4)\right)\le C < X_0R(T), & \text{if } \|\theta\|_T \ge \frac{\phi(|z_\alpha|)}{\alpha} \\ 0\le C < X_0R(T), & \text{if } \|\theta\|_T < \frac{\phi(|z_\alpha|)}{\alpha}. \end{cases}$$

The portfolio that maximizes $\mathbb{E}[X^\pi(T)]$ under the constraint $\mathrm{CCaR}\big(X^\pi(T)\big)\le C$ is

$$\pi_{\epsilon_5}=\frac{\epsilon_5}{\|\theta\|_T}\pi_M(t),$$

where ϵ_5 solves

$$\epsilon_5\,\|\theta\|_T+\ln\Phi(-|z_\alpha|-\epsilon_5)-\ln\alpha-c=0,$$

where

$$c=\ln\left(1-\frac{C}{X_0R_0(T)}\right).$$

The corresponding expected wealth is

$$\mathbb{E}\left[X^{\pi_{\epsilon_5}}(T)\right]=X_0R_0(T)\mathrm{e}^{\epsilon_5\|\theta\|_T}.$$

The setting

- **Bond process:**

$$dS_0(t) = rS_0(t)dt, \qquad S_0(0) = s_0.$$

Then, with r constant, $S_0(t) = s_0 e^{rt}$.

- **Stock/commodity processes:**

$$\frac{dS_i(t)}{S_i(t)} = \beta_i(L_i - c_i \ln S_i(t))dt + \sum_j \sigma_{ij} dW_j(t), \quad i = 1, \ldots, m.$$

- Define $\boldsymbol{W}(t) := \big(W_1(t), \ldots, W_m(t)\big)'$, $a_i := \beta_i L_i$, $b_i := \beta_i c_i$ and $\boldsymbol{\sigma_i} := (\sigma_{i1}, \ldots, \sigma_{im})$. Then

$$\frac{dS_i(t)}{S_i(t)} = (a_i - b_i \ln S_i(t))dt + \boldsymbol{\sigma_i} d\boldsymbol{W}(t), \quad i = 1, \ldots, m.$$

Log transform

- Let $Y_i(t) := \ln S_i(t)$ so that

$$dY_i(t) = (\hat{a}_i - b_i Y_i(t))dt + \boldsymbol{\sigma_i} d\boldsymbol{W}(t), \quad i = 1, \ldots, m,$$

where $\hat{a}_i = a_i - \frac{1}{2} \|\boldsymbol{\sigma_i}\|^2$.

- We can use this expression to compute the means and covariances of the normally-distributed random variable $Y_i(t)$:

$$\mathbb{E}[Y_i(t)] = Y_i(0)\mathrm{e}^{-b_i t} + \hat{a}_i \mathcal{E}(t, b_i)$$

and

$$\mathrm{Cov}[Y_i(t), Y_j(t)] = \boldsymbol{\sigma_i} \boldsymbol{\sigma_j}' \mathcal{E}(t, b_i + b_j),$$

where

$$\mathcal{E}(t, b) := \int_0^t \mathrm{e}^{-b\tau} d\tau.$$

Solving the SDEs

The substitution

$$Z_i(t) = \mathrm{e}^{b_i t} Y_i(t)$$

yields

$$dZ_i(t) = \mathrm{e}^{b_i t} \hat{a}_i dt + \mathrm{e}^{b_i t} \boldsymbol{\sigma_i} d\boldsymbol{W}(t),$$

i.e.,

$$Z_i(t) = z_{i,0} + \hat{a}_i \mathcal{E}(t, -b_i) + \int_0^t \mathrm{e}^{b_i u} \boldsymbol{\sigma_i} d\boldsymbol{W}(u).$$

This leads to

$$Y_i(t) = \mathrm{e}^{-b_i t} y_{i,0} + \hat{a}_i \mathcal{E}(t, b_i) + \mathrm{e}^{-b_i t} \boldsymbol{\sigma_i} \int_0^t \mathrm{e}^{b_i u} d\boldsymbol{W}(u),$$

which in turn will provide the means and covariances.

Wealth process

- $X(t) = \sum_{i=0}^{m} N_i(t) S_i(t)$.
- Portfolio: set $\pi_i := \frac{N_i(t)S_i(t)}{X(t)} \in \mathbb{R}$, so that $\sum_i \pi_i = 1$, and let $N_i(t)$ be such that $\pi_i =$constant.
- $\boldsymbol{N}(t)$ is a self-financing strategy, so that

$$
\begin{aligned}
dX(t) &= \sum_{i=0}^{m} N_i(t) dS_i(t) \\
&= N_0 S_0(t) r dt \\
&\quad + \sum_{i=1}^{m} N_i(t) S_i(t) \left[(a_i - b_i \ln S_i(t)) dt + \boldsymbol{\sigma_i} d\boldsymbol{W}(t) \right] \\
&= X(t) \left(r dt + \sum_{i=1}^{m} \pi_i \left[(a_i - r - b_i \ln S_i(t)) dt + \boldsymbol{\sigma_i} d\boldsymbol{W}(t) \right] \right).
\end{aligned}
$$

Log-wealth process

Set $H(t) := \ln X(t)$. Then

$$dH(t) = \mu dt + \sum_i \pi_i \left[-b_i Y_i(t)dt + \boldsymbol{\sigma_i} d\boldsymbol{W}(t)\right],$$

where

$$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)',$$

$$\sigma = \begin{pmatrix} \boldsymbol{\sigma_1} \\ \vdots \\ \boldsymbol{\sigma_m} \end{pmatrix},$$

and

$$\mu = r + \boldsymbol{\pi}'(\boldsymbol{a} - r\mathbf{1}) - \frac{1}{2}\left\|\boldsymbol{\pi}'\sigma\right\|^2,$$

with $\boldsymbol{a} = (a_1, \ldots, a_m)'$.

Log-wealth

From the previous slide,

$$dH(t) = \mu dt + \sum_i \pi_i \left[-b_i Y_i(t) dt + \boldsymbol{\sigma_i} d\boldsymbol{W}(t) \right].$$

Turning our attention to the terms in square brackets, we find

$$-b_i Y_i(t) dt + \boldsymbol{\sigma_i} d\boldsymbol{W}(t) = dY_i(t) - \hat{a}_i dt.$$

Thus

$$dH(t) = (\mu - \boldsymbol{\pi}' \hat{\boldsymbol{a}}) dt + \boldsymbol{\pi}' d\boldsymbol{Y}(t),$$

and so

$$H(t) = H(0) + (\mu - \boldsymbol{\pi}' \hat{\boldsymbol{a}}) t + \boldsymbol{\pi}' (\boldsymbol{Y}(t) - \boldsymbol{Y}(0)).$$

Mean and variance

From our knowledge of the mean and variance of $\boldsymbol{Y}$ we can thus deduce

$$\mathbb{E}[H(t)] = H(0) + (\mu - \boldsymbol{\pi}'\hat{\boldsymbol{a}})t + \sum_i \pi_i \mathcal{E}(t, b_i)(\hat{a}_i - b_i Y_i(0))$$

and

$$\mathbb{V}[H(t)] = \boldsymbol{\pi}' F(t, \boldsymbol{b})\boldsymbol{\pi},$$

where the positive-definite matrix F (at least it is for $t > 0$) has entries

$$F_{ij}(t, \boldsymbol{b}) = \boldsymbol{\sigma_i}\boldsymbol{\sigma_j}'\mathcal{E}(t, b_i + b_j).$$

Note that if we set $\boldsymbol{\mathcal{F}}(t, \boldsymbol{b})$ to be the vector with entries $\mathcal{F}_i = \mathcal{E}(t, b_i)(\hat{a}_i - b_i Y_i(0)) - \hat{a}_i t$, then $\boldsymbol{\mathcal{F}}(t, \boldsymbol{0}) = \boldsymbol{0}$ and

$$\mathbb{E}[H(t)] = H(0) + \mu t + \boldsymbol{\pi}'\boldsymbol{\mathcal{F}}(t, \boldsymbol{b}).$$

Quantiles

We denote the α-quantile of a random variable X by q_X, and we have

$$\begin{aligned} q_{H(t)} &= \mathbb{E}[H(t)] - |z_\alpha| \sqrt{\mathbb{V}[H(t)]} \\ &= H(0) + rt + \boldsymbol{\pi}' \left[(\hat{\boldsymbol{a}} - r\mathbf{1})t + \boldsymbol{\mathcal{F}}(t, \boldsymbol{b}) \right] \\ &\quad - \frac{t}{2} \left\| \boldsymbol{\pi}' \sigma \right\|^2 - |z_\alpha| \sqrt{\boldsymbol{\pi}' F(t, \boldsymbol{b}) \boldsymbol{\pi}}. \end{aligned}$$

We set $\boldsymbol{g}(t) := (\hat{\boldsymbol{a}} - r\mathbf{1})t + \boldsymbol{\mathcal{F}}(t, \boldsymbol{b})$ so that

$$f(\boldsymbol{\pi}, t) = \boldsymbol{\pi}' \boldsymbol{g}(t) - \frac{t}{2} \left\| \boldsymbol{\pi}' \sigma \right\|^2 - |z_\alpha| \sqrt{\boldsymbol{\pi}' F(t, \boldsymbol{b}) \boldsymbol{\pi}}.$$

Then we can write the α-quantile of the wealth process X as

$$q_{X(t)} = X(0) \mathrm{e}^{rt} \mathrm{e}^{f(\boldsymbol{\pi}, t)}.$$

Capital-at-Risk (CaR) and Value-at-Risk (VaR)

- Recall that CaR is the difference between the riskless wealth and the α-quantile:

$$\mathrm{CaR} = X(0)\mathrm{e}^{rt}\left(1 - \mathrm{e}^{f(\boldsymbol{\pi},t)}\right).$$

- VaR is the difference between the expected wealth and the α-quantile:

$$\mathrm{VaR} = X(0)\mathrm{e}^{rt+\boldsymbol{\pi}'\boldsymbol{g}(t)-\frac{t}{2}\|\boldsymbol{\pi}'\sigma\|^2} \times \left\{\mathrm{e}^{\frac{1}{2}\boldsymbol{\pi}'F(t,\boldsymbol{b})\boldsymbol{\pi}} - \mathrm{e}^{-|z_\alpha|\sqrt{\boldsymbol{\pi}'F(t,\boldsymbol{b})\boldsymbol{\pi}}}\right\}.$$

Minimizing CaR

We seek to minimize CaR $=$ CaR$(\boldsymbol{\pi}, T)$ at the time horizon T.
Note that

$$\begin{aligned}\operatorname*{argmin}_{\boldsymbol{\pi}} \mathrm{CaR}(\boldsymbol{\pi}, T) &= \operatorname*{argmax}_{\boldsymbol{\pi}} f(\boldsymbol{\pi}, T) \\ &= \operatorname*{argmax}_{\boldsymbol{\pi}} \boldsymbol{\pi}' \boldsymbol{g}(T) - \frac{T}{2} \left\| \boldsymbol{\pi}' \sigma \right\|^2 - |z_\alpha| \sqrt{\boldsymbol{\pi}' F \boldsymbol{\pi}}.\end{aligned}$$

If we look for critical points in this expression, we certainly have one when $\boldsymbol{\pi} = \mathbf{0}$. Also, we find that the gradient is zero when

$$\boldsymbol{g}(T) - T\sigma\sigma'\boldsymbol{\pi} - \frac{|z_\alpha|}{\sqrt{\boldsymbol{\pi}' F \boldsymbol{\pi}}} F\boldsymbol{\pi} = \mathbf{0}.$$

But this implies that

$$\boldsymbol{\pi} = \left(T\sigma\sigma' + \frac{|z_\alpha|}{\sqrt{\boldsymbol{\pi}' F \boldsymbol{\pi}}} F \right)^{-1} \boldsymbol{g}(T).$$

Minimizing CaR

If we write $\lambda = \sqrt{\boldsymbol{\pi}' F \boldsymbol{\pi}}$, then we find that λ must satisfy

$$\left\| A \left(\lambda T \sigma \sigma' + |z_\alpha| \, F \right)^{-1} \boldsymbol{g}(T) \right\| = 1,$$

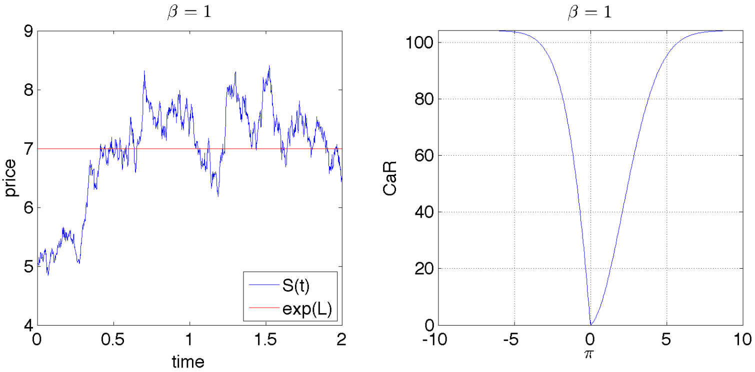
where $F = A'A$.
This equation can be shown to have a unique positive solution if $\boldsymbol{g}(T)' F^{-1} \boldsymbol{g}(T) > |z_\alpha|^2$. In this case the optimimum $\boldsymbol{\pi}$ is given by

$$\boldsymbol{\pi} = \left(T \sigma \sigma' + \frac{|z_\alpha|}{\lambda} F \right)^{-1} \boldsymbol{g}(T).$$

Otherwise the optimum portfolio has $\boldsymbol{\pi} = \mathbf{0}$.

One-dimensional results

Typical asset paths (left) and 2-year CaR (right) for the one-dimensional asset model with parameters: $L = \log 7, c = 1, S(0) = 5, \sigma = 0.3, T = 2, X_0 = 100$, and for $\beta = 1, 2, 5$ and 50.



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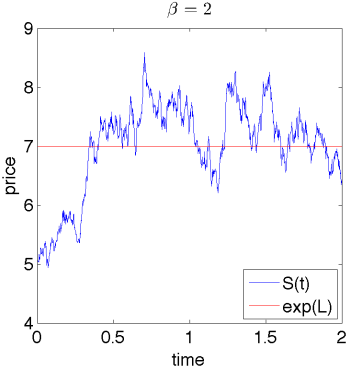

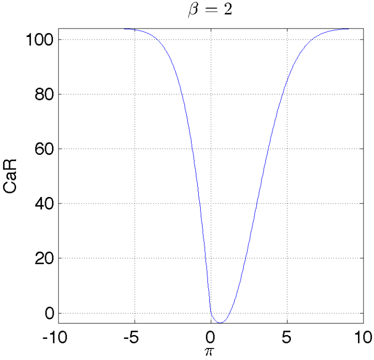

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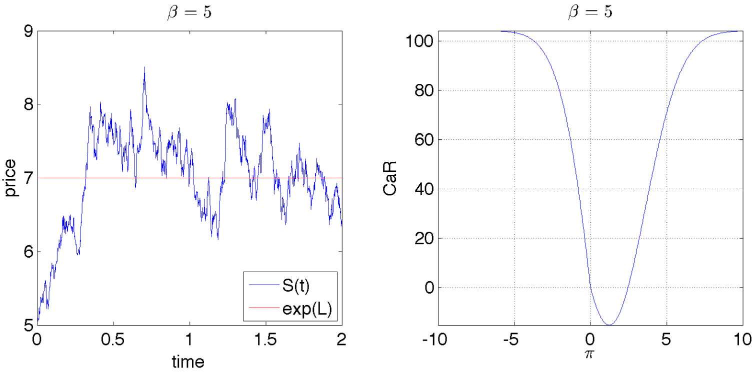

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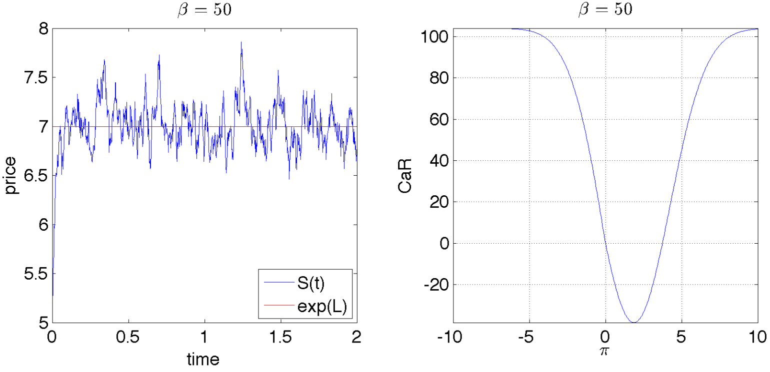
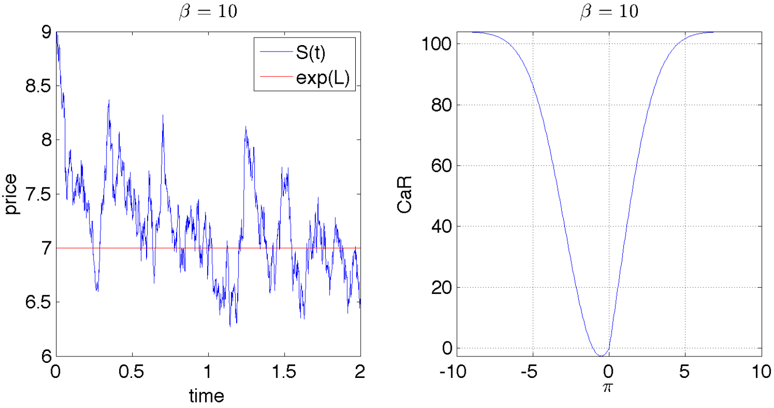


One-dimensional results

Typical asset path (left) and 2-year CaR (right) for the one-dimensional asset model with parameters: $L = \log 7, c = 1, S(0) = 9, \sigma = 0.3, T = 2, X_0 = 100$, and for $\beta = 10$.



Multi-asset results

We assume that the correlation matrix is $\boldsymbol{\rho} = \begin{bmatrix} 1.0 & -0.6 & 0.5 \\ -0.6 & 1.0 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$.

The final time $T = 1$, $X_0 = 100$, and $r = 0.02$; $\alpha = 0.05$ and the other coefficients, and the resulting optimal strategy (π_1, π_2, π_3), are given by

$\boldsymbol{\pi}$	-1.02	-1.65	2.35
$\boldsymbol{L}$	0.05	0.50	1.50
$\boldsymbol{\beta}$	1	2	5
$\boldsymbol{Y(0)}$	1	1	1

CaR as a function of π