The Volatility Surface (Jim Gatheral)

Chapter 2: The Heston Model

Tony Ware
University of Calgary

5th October, 2010

The process

The Heston model (Heston (1993))

$$
\begin{aligned}
dS_t &= \mu_t S_t dt + \sqrt{v_t} S_t dZ_t^1 \\
dv_t &= -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t} dZ_t^2
\end{aligned}
$$

with $\langle dZ_t^1, dZ_t^2 \rangle = \rho dt$.
This is an affine process (Duffie *et al.* (2000)): the drifts and covariance are linear in S_t and v_t.

The valuation equation

$$
\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \eta v S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\eta^2 v \frac{\partial^2 V}{\partial v^2} + rS\frac{\partial V}{\partial S} - rV = \lambda(v - \bar{v})\frac{\partial V}{\partial v}.
$$

Note that Gatheral has assumed that the original process is already in the pricing measure.

European option pricing

The valuation equation rewritten

- Consider a call option expiring at T with strike price K.
- Set $x = \ln \frac{F_{t,T}}{K}$, where $F_{t,T}$ is the T-forward price of the stock index at time t.
- Set $\tau = T - t$.
- Write $C(\tau, x, v) = \mathrm{e}^{r\tau} V(t, S, v)$.

$$-C_\tau + \frac{1}{2} v C_{xx} - \frac{1}{2} v C_x + \frac{1}{2} \eta^2 v C_{vv} + \rho \eta v C_{xv} - \lambda (v - \overline{v}) C_v = 0$$

with final time condition $C(0, x, v) = K(\mathrm{e}^x - 1)_+$.

According to Duffie *et al.* (2000), we can look for the solution in the form

$$C(\tau, x, v) = K\left(\mathrm{e}^x P_1(\tau, x, v) - P_0(\tau, x, v)\right).$$

European option pricing

P equations: for $j = 0, 1$

$$-\frac{\partial P_j}{\partial \tau} + \frac{v}{2}\frac{\partial^2 P_j}{\partial x^2} - (-1)^j \frac{v}{2}\frac{\partial P_j}{\partial x} + \eta^2 \frac{v}{2}\frac{\partial^2 P_j}{\partial v^2} + \rho \eta v \frac{\partial^2 P_j}{\partial v \partial x} + (a - b_j v)\frac{\partial P_j}{\partial v} = 0$$

where $a = \lambda \bar{v}$ and $b_j = \lambda - j\rho\eta$.

Final time conditions: $\lim_{\tau \to 0} P_j(\tau, x, v) = \mathbf{1}_{\{x>0\}}$.

Fourier transforms

Set, for $j = 0, 1$,

$$\tilde{P}_j(\tau, u, v) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} P_j(\tau, x, v)\,dx,$$

so that

$$P_j(\tau, x, v) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux} \tilde{P}_j(\tau, u, v)\,du.$$

European option pricing

Transformed equations

The P equations have constant coefficients in x, and transform into

$$v\left[\alpha_j \tilde{P}_j - \beta_j \frac{\partial \tilde{P}_j}{\partial v} + \gamma \frac{\partial^2 \tilde{P}_j}{\partial v^2}\right] + a \frac{\partial \tilde{P}_j}{\partial v} - \frac{\partial \tilde{P}_j}{\partial \tau} = 0,$$

where

$$\begin{aligned} \alpha_j &= -\frac{u^2}{2} - \frac{\mathrm{i}u}{2} + \mathrm{i}ju \\ \beta_j &= \lambda - \rho\eta j - \rho\eta \mathrm{i}u \\ \gamma &= \frac{\eta^2}{2}. \end{aligned}$$

We look for solutions of this in the form

$$\tilde{P}_j(\tau, u, v) = \mathrm{e}^{C_j(\tau,u)\bar{v} + D_j(\tau,u)v}\,\tilde{P}_j(0, u, v).$$

European option pricing

The result is a set of Riccati equations for C_j and D_j:

$$\frac{\partial C_j}{\partial \tau} = \lambda D_j; \qquad \frac{\partial D_j}{\partial \tau} = \gamma (D_j - r_j^+)(D_j - r_j^-),$$

where

$$r_j^\pm = \frac{\beta_j \pm \sqrt{\beta_j^2 - 4\alpha_j \gamma}}{2\gamma} =: \frac{\beta_j \pm d_j}{\eta^2}.$$

Given that $C_j(0,u) = D_j(0,u) = 0$, Gatheral deduces

$$D(\tau,u) = r_j^- \frac{1 - \mathrm{e}^{-d_j\tau}}{1 - g_j \mathrm{e}^{-d_j\tau}}; \quad C(\tau,u) = \lambda r_j^- \tau - \frac{2\lambda}{\eta^2} \ln\left(\frac{1 - g_j \mathrm{e}^{-d_j\tau}}{1 - g_j}\right),$$

where $g_j = r_j^- / r_j^+$.

European option pricing

The final step…

…is to compute

$$P_j(\tau, x, v) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left\{ \frac{\mathrm{e}^{C_j(\tau,u)\bar{v} + D_j(\tau,u)v + \mathrm{i}ux}}{\mathrm{i}u} \right\} du,$$

and compute the option value via

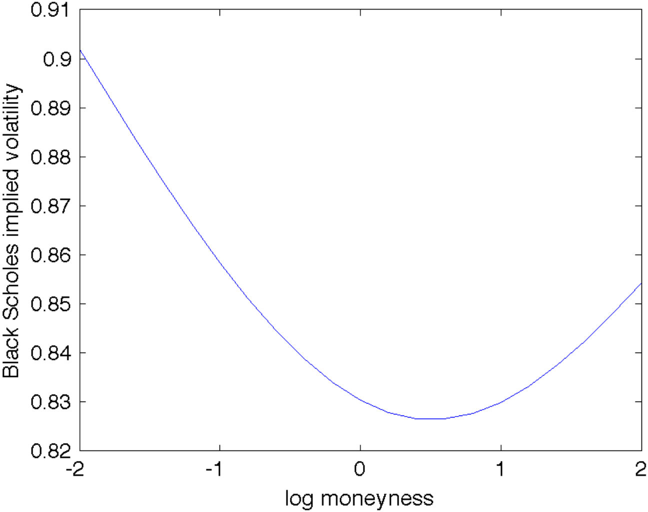
$$C(\tau, x, v) = K\left(\mathrm{e}^x P_1(\tau, x, v) - P_0(\tau, x, v)\right).$$

Two comments

- The complex logarithm in the formula for C_j should be continuous with respect to u. This appears to always be the case when the principal value is used in this formulation.
- The Heston conditional characteristic function can also be computed using the functions C_j and D_j.

European option pricing

Black Scholes implied volatilities from Heston option prices



Simulation

The SDE again

$$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dZ_t^1$$
$$dv_t = -\lambda(v_t - \bar{v})dt + \eta\sqrt{v_t} dZ_t^2$$

with $\langle dZ_t^1, dZ_t^2 \rangle = \rho dt$.

Euler discretization

The variance process becomes

$$v_{i+1} = v_i - \lambda(v_i - \bar{v})\Delta t + \eta\sqrt{v_i \Delta t} Z.$$

The convergence is extremely slow.

Milstein discretization

Incorporating higher order terms in the Itô-Taylor expansion leads to

$$\begin{aligned} v_{i+1} &= v_i - \lambda(v_i - \bar{v})\Delta t + \eta\sqrt{v_i \Delta t} Z + \frac{\eta^2}{4}\Delta t(Z^2 - 1) \\ &= \left(\sqrt{v_i} + \frac{\eta}{2}\sqrt{\Delta t} Z\right)^2 - \lambda(v_i - \bar{v})\Delta t - \frac{\eta^2}{4}\Delta t. \end{aligned}$$

The stock process should be discretized in terms of $x_i = \log S_i/S_0$: $x_{i+1} = x_i - \frac{v_i}{2}\Delta t + \sqrt{v_i \Delta t} W$, where $\mathbb{E}[ZW] = \rho$.

References

Duffie, Darrell, Pan, Jun, & Singleton, Kenneth. 2000. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, **68**(6), 1343–1376.

Heston, Steve L. 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, **6**(2), 327–343.