Affine General Equilibrium Models

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Outline

1. Introduce the recursive utility function
2. Optimization problem
3. Euler Equation

Recursive Utility function (Koopmans (1960))

$$V(c_0, c_1, \ldots) = W\left(c_0, V(c_1, c_2, \ldots)\right)$$

Here, the recursive utility function has deterministic consumption streams.
In the stochastic case, the consumption stream is stochastic, and the future utility is random. Therefore, we need to compute a certainty equivalent for random future utility.
Certainty equivalent functionals

$$\mu : dom\ \mu \subseteq M(R_+) \rightarrow R_+,$$

where $M(R_+)$ is the space of Borel probability measure,

$$\mu(\delta_x) = x \subseteq R_+$$

Elasticity of substitution

What is Elasticity of Substitution?

Elasticity of substitution is the ratio of percentage change in production (or utility) function to the percentage change in their marginal products (or utilities).

The elasticity of substitution of a function of two variables

If $f(x_1, x_2)$ is homogeneous of some degree k and strictly quasi-concave,then $f_1(x_1, x_2)/f_2(x_1, x_2)$ is determined by x_1/x_2 and it is a decreasing function of x_1/x_2.

Elasticity of substitution

If f is homogeneous of degree k, we get $f(\lambda x) = \lambda^k f(x)$.

$$\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{x_2^{k-1} f_1(\frac{x_1}{x_2}, 1)}{x_2^{k-1} f_2(\frac{x_1}{x_2}, 1)} = \frac{f_1(\frac{x_1}{x_2}, 1)}{f_2(\frac{x_1}{x_2}, 1)}$$

the elasticity of substitution function is

$$\sigma(\frac{p_1}{p_2}) = -\frac{\frac{p_1}{p_2} h'(\frac{p_1}{p_2})}{h(\frac{p_1}{p_2})} = -\frac{d \ln h(\frac{p_1}{p_2})}{d \ln(\frac{p_1}{p_2})}$$

$$\frac{1}{\sigma(p_1/p_2)} = -\frac{d \ln \frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}}{d \ln \frac{x_1}{x_2}}$$

If the constant elasticity of substitution $\sigma = 1/(1-\rho)$, the production function follows the form of

$$f(x_1, \ldots, x_n) = A\Big(\sum_{i=1}^{n} \lambda_i x_i^{\rho}\Big)^{k/\rho}$$

where $A > 0$, $k > 0$, $\lambda_i \geq 0$ for all i, $\sum_i \lambda_i = 1$.

Aggregator function

In order to make existence of recursive utility function, we restrict the aggregator function W has the CES form,

$$W(c,z) = [c^\rho + \beta z^\rho]^{1/\rho}$$

Where $0 \neq \rho < 1,\ 0 < \beta < 1$

- the constant elasticity of substitution $\sigma = 1/(1-\rho)$
- z rerpesents certainty equivalent future utility

Recursive utility function

In the stochastic case, we define the recursive utility function on its domain as follows:

$$V(c_0, m) = W(c_0, \mu(V[m]))$$

for some aggregator function W:

$$R_+^2 \rightarrow R_+$$

and some certainty equivalent μ.

- m represents the probability distribution of future consumption.
- V[m] represents the probability measure for future utility.

Examples of certainty equivalent functionals μ

Power Utility

$$\mu(p) \equiv \Big(\int x^{\rho} dp(x) \Big)^{1/\rho} \equiv (E_p \tilde{x}^{\rho})^{1/\rho}, \quad p \in M(R_+)$$

where $0 \neq \rho < 1$, p is $V[m]$ in recursive utility function and x denotes the future utility.

Kreps/Porteus

$$\mu(p) \equiv \left(E_p \tilde{x}^{\alpha} \right)^{1/\alpha}, \; p \in M(R_+)$$

where $0 \neq \alpha < 1$, V satisfies the recursive relation

$$V(c_0, m) = [c_0^{\rho} + \beta(E_p V^{\alpha})^{\rho/\alpha}]^{1/\rho}$$

There are two separated parameters: risk aversion and elasticity of substitution.

Existence of recursive utility

Let $W(c,z) = [c^\rho + \beta z^\rho]^{1/\rho}$, and let μ be a mean value functional satisfying some "mild" assumptions, there exists a solution $V(c_0, m) = W(c_0, \mu(V[m]))$.

Assumption on mean value functionals

MV.1:*continuity* if p_n and p are in $M([0,a]) \subset M(R_+)$, then

- $lim \int fdp_n = \int fdp \; \forall f : R_+ \to R_+$
  $\Rightarrow \lim \mu(p_n) = \mu(p)$, and
- $\limsup \int fdp_n \leqslant \int fdp$ for all f: $R_+ \to R_+$
  $\Rightarrow \limsup \mu(p_n) \leqslant \mu(p)$

MV.2:*homogeneity* $\mu(p_{\lambda\bar{x}}) = \lambda\mu(p_{\bar{x}})$ for all > 0, where $p_{\lambda\bar{x}}$ and $p_{\bar{x}}$ are probability measures in dom μ corresponding to the random variables $\bar{x}$ and $\lambda\tilde{x}$ respectively.

We want to solve optimization problem:

$$J(x_0, I_0) = \sup(V(h))$$

over all feasible plans $h(t)$
We determine the optimal consumption and portfolio behavior of an individual who faces a exogenous rates of return.
We denote a plan $h(t)$ is sequence of (x_t, I_t). The interpretation of $h_t(x_t, I_t) = (c_t, \omega_t)$

- x_t represents the aggregate wealth
- c_t represents the consumption stream
- ω_{kt} represents the investment proportion in kth asset, where $\omega_t = (\omega_1, \ldots, \omega_{kt})'$

Optimal plans

Homogeneous plan A plan is *homogeneous* if $\forall t > 0$ and $\forall (x, I_t) \in \omega_t,\ h_t(1, I_t) = (c_t, \omega_t) \Rightarrow h_t(x, I_t) = (c_t x, \omega_t)$. Because of the homotheticity of preferences and the linearity of "technology," it is natural to restrict oneself to homogeneous plans which are henceforth simply plans.

Feasible plan A plan is *feasible* if $\forall t \geq 0$ and $\forall (x_t, I_t) \in \Omega_t$, $c_t \leq x_t$ where c_t is the first component of $h_t(x_t, I_t)$ and where wealth evolves according to

$$x_t = (x_{t-1} - c_{t-1})\omega_{t-1}^{j}\tilde{r}_{t-1},\ \ t \geq 1, x_0 \geq 0.$$

where ω_{t-1}^{j} is fraction of wealth investment in the *jth* investment of $\omega_1 \ldots, \omega_k$, r_{t-1} is the return in period t, and $(r_t)_{t \geqslant 0}$ is a stochastic process

Model

We start with a discrete time formation of real endowment economy where the investors' preferences over the uncertain consumption stream C_t can be described by a recursive utility function of Kreps and Porteus (1978), Epstein and Zin (1989) and Wei (1989):

$$U_t = [(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t U_{t+1}^{1-\gamma})^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}}$$

The representative agent's preferences are thus characterized by a discount factor δ, the elasticity of substitution ψ, local risk aversion coefficient γ, and $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$.

Euler equation

The optimal consumption and wealth allocation vector $w*$ satisfies Euler equation.

$$E_t\left[\delta^{\theta}\left(\frac{C_{t+1}}{C_t}\right)^{\frac{\theta}{\psi}}R_{c,t+1}^{-(1-\theta)}R_{i,t+1}\right]=1$$

- $R_{c,t}$ is the return on the aggregate wealth portfolio
- $R_{i,t}$ is the return on an arbitrary asset

Assumption

X is state variable, which follows the affine jump diffusion process. There are additional joint restrictions on the parameters of the model, We assume that the log consumption and dividend growth rates are linear in the states:

$$d \ln C_t = \delta_c' dX_t$$

$$d \ln D_t = \delta_d' dX_t$$

δ_c *and* δ_d are selection vectors $(1, 0, 0, \ldots, 0)$ and $(0, \ldots, 0, 0, 1)$, respectively.

Assumption

Suppose X_t is a Markov process with a stochastic differential equation representation

$$dX_t = \mu(X_t)dt + \Sigma(X_t)dW_t + \xi_t dN_t$$

Assume the moment generation function of jump size exists

$$Ee^{\mu\xi} = \rho(u)$$

So ρ is well-defined in both real and complex numbers.

$$\mu(X_t) = \mathcal{M} + \mathcal{K}X_t$$

$$\Sigma(X_t)\Sigma(X_t)' = h + \sum_i H_i X_{t,i}$$

$$\ell(X_t) = \ell_0 + \ell_1 X_t$$

where $(\mathcal{M}, \mathcal{K}) \in R^n \times R^{n\times n}$, $(h, H) \in R^{n\times n} \times R^{n\times n\times n}$, $(\ell_0, \ell_1) \in R^n R^{n\times n}$

Equilibrium

- Deriving the equilibrium pricing kernel in our economy in continuous time
- The strategy is to translate the Euler condition in discrete time into the martingale restriction in continuous time.
- Setting $R_{i,t+1} = R_{c,t+1}$ in Euler equation, solve for the equilibrium return on the aggregate wealth portfolio.
- Characterizing the pricing kernel and the risk-neutral probability measure can be used to price any asset in the economy.

Equilibrium

We define the discrete time return to be the return on a portfolio which re-invests the continuously paid dividends. The discrete time return on this asset is just the aggregate continuous time log return,

$$\int_t^{t+1} d\ln R_{i,s}$$

The Euler equation becomes

$$E_t exp\left[\ln \frac{M_{t+1}}{M_t} + \int_t^{t+1} d\ln R_{i,s}\right] = 1$$

where M_t is the marginal utility of the agents, whose log-increments in discrete time are given by

$$\ln M_{t+1} - \ln M_t = \theta \ln \delta - \frac{\theta}{\psi}(_{t+1} - \ln C_t) - (1-\theta)\int_t^{t+1} d\ln R_{c,s}$$

Pricing kernel

Price of any instrument that pays a divident or a payoff at time T let

$$P_t = E_P[M_T P_T | F_t]$$

$(M_T)_{T \geqslant 0}$ is the pricing kernel.

$$M_t = \frac{dQ_t}{dP_t} e^{-\int_t^T r_s ds}$$

Under risk neutral measure we calculate $P_t = E_Q[e^{-\int_t^T r_s ds} P_T]$ In Euler Equation, we can see that
$E_Q[R_{c,t+1} e^{-\int_t^T r_s ds} | F_t] = E_P[\frac{M_{t+1}}{M_t} R_{c,t+1} | F_t] = 1$
For example:
$B(0,t) = E_Q[e^{-\int_0^t r_s ds} 1] = E_P[M_t 1]$

Pricing kernel

Euler equation gives the form of the pricing kernel:

$$d\ln M_t = \theta \ln \delta dt - \frac{\theta}{\psi} d\ln C_t + (\theta - 1) d\ln R_{c,t}$$

$$= (\theta \ln \delta - (\theta - 1)\ln k_1 + (\theta - 1)(k_1 - 1)B'(X_t - \mu_X))dt - \lambda' dX_t$$

The author used a linearization techuique to solve k_0 and k_1, which is the value choosing in the approximation.

Linearize the discrete time model

We follow Campbell and Shiller (1988), Campbell (1993) and Bansal and Yaron (2004), among others, to linearize the model. The discrete time continuously compound (log) return $\ln R_t$

$$\ln R_{t+1} = \ln \frac{P_{t+1} + D_{t+1}}{P_t} \equiv \ln(e^{\frac{P_{t+1}}{D_{t+1}}} + 1) - \ln \frac{P_t}{D_t} + \ln \frac{D_{t+1}}{D_t}$$

Log-linearize the first summand around the mean log price-dividend ratio to obtain

$$\ln R_{t+1} \approx k_0 + k_1 v_{t+1} - v_t + \Delta \ln D_{t+1}$$

Linearize the discrete time

The constant k_0 and k_1 depend on the mean log valuation ratio $E(v_t)$. The approximation error is given by the second-order Tylor residual

let $x* = E[v_t] = \log -(\frac{k_1}{k_1-1})$, then compute

$$f(x) = \ln(e^{\ln(-\frac{k_1}{k_1-1})} + 1) + \frac{e^{\ln(-\frac{k_1}{k_1-1})}}{e^{\ln(-\frac{k_1}{k_1-1})} + 1}(x - \ln(-\frac{k_1}{k_1 - 1}))$$

$$= \ln(\frac{1}{1 - k_1}) - k_1 \ln(\frac{k_1}{1 - k_1}) + k_1 x$$

$$k_1 = \frac{e^{E(v_t)}}{1 + e^{E(v_t)}}$$

$$k_0 = -\ln[(1 - k_1)^{1-k_1} k_1^{k_1}]$$

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