# Some observations on implied volatility

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## Plan of the talk

- Review of standard approach.
- Review of Gram-Charlier expansions.
- Main results - connection between the pricing density and volatility skew.

## Implied volatility

Under the Black-Scholes model (or more accurately Black's model), the premium for a European call on futures is given by

$$\begin{aligned} &C_{BS}(K,\sigma) \\ &\quad = e^{-rT}[F\Phi(d_1(K,\sigma)) - K\Phi(d_2(K,\sigma))], \quad (1) \end{aligned}$$

where

$$d_1(K,\sigma) = \frac{\log(\frac{F}{K})}{\sigma T^{1/2}} + \frac{\sigma T^{1/2}}{2}, \quad (2)$$

$$d_2(K,\sigma) = \frac{\log(\frac{F}{K})}{\sigma T^{1/2}} - \frac{\sigma T^{1/2}}{2}, \quad (3)$$

$\exp(-rT)$ is the discount factor, and $\sigma$ is the volatity parameter. ($\Phi$ and $\phi$ will denote the cdf and the pdf of the standard normal distribution, respectively.) Note the variance of the (normally distributed) log-return on the futures contract is

$$\text{var}\left(\log\left(\frac{F_T}{F}\right)\right) = \sigma^2 T. \quad (4)$$

The reality has been very well documented not to be Black-Scholes, and observed option premiums do not agree with (1). Clearly, for a given $K$, a value $\sigma_K$ can always be found so that (1) holds for that strike, but in practice these values will differ across different strikes. The quantity $\sigma_K$ thus obtained is called the **implied volatility**.

*" . . . implied volatility is the wrong number to put in the wrong formula to obtain the right price."*

- Ricardo Rebonato *Volatility and Correlation in Option Pricing*, Wiley, 1999.

The practice of estimating $\sigma_K$ in conjunction with Black-Scholes pricing is so widespread that one can fully expect to be performed daily on every option desk.

- Hull observes (pg. 336, Section 15.4 *Options, Futures & Other Derivatives, 5 edition*), that traders and analysts tend to parametrise the implied volatility surface by
$$m_H(K) = \frac{1}{\sqrt{T}} \log \frac{K}{F}, \tag{5}$$
rather than by $K$ directly.

- The author's personal experience largely confirms this observation: for a given expiry $T$, most practitioners appear to parametrise the volatiltiy curve by
$$m(K) = \log \frac{K}{F} \tag{6}$$

or something similar to it, then fit some parametric or non-parametric curve to the data.

- Another commonly encountered approach rests on the notion of "delta equivalence." The delta in the Black-Scholes model is defined as

  $$\frac{\partial}{\partial F} C_{BS}(K, \sigma) = e^{-rT} \Phi(d_1(K, \sigma)). \qquad (7)$$

  Delta-equivalent implied volatility curve is parametrised by delta. Ignoring the discounting, this is equivalent to parametrising the curve by $d_1$, which is a linear fuction of $m$ in (5). We could thus define

  $$m_\delta(K) = d_1(K, \sigma). \qquad (8)$$

- Once the decision is made regarding the use of $m$, $m_H$, $m_\delta$, or yet another function of $K$, a function $\alpha(m)$ if obtained from available market premium quotes, where typically $\alpha$ is the correction term in the implied volatility with respect to the market implied volatility $\sigma_{ATM}$ for at-the-money options, $K = F$ (or $m = 0$). Very often the curve $\alpha$ is modelled as a second-degree polynomial or higher, in $m$. The chosen function $m$, $m_H$, $m_\delta$, or some other variation is sometimes called the **moneyness** function. **The important point for the discussion in this paper is that moneyness is constant in** $\log(K/F)$**.**

- It is then assumed that as the market price $F$ chages (and potentially as $\sigma_{ATM}$ changes), the function $\alpha$ — usually called the skew — remains unchanged. If the price moves from $F$ to $F'$ and a trader wants to price an option with strike $K$, she will use $\sigma' =$

$\sigma_{ATM} + \alpha(m(K))$ and plug it into (1) along with $F'$. (Note that $m(K)$ is now calculated with respect to the new price $F'$.)

- While there seems to be some empirical evidence that thus constructed skew curves do remain constant over fairly long periods of time (at least in energy, equity and FX markets), the whole approach feels quite *ad hoc*, and little, if any, theoretical justification has been offered for it.

- The connection between the skew function and the underlying process dynamics seems indirect and lacking in transparency.

## Hermite polynomials and normal density

Recall: Hermite polynomials are orthogonal with respect to the standard normal density. With $H_0(x) = 1$ and $H_1(x) = x$, they possess the following properties:

$$\begin{aligned} H_{n+1}(x) &= xH_n(x) - nH_{n-1}(x) && (9) \\ \frac{d}{dx}H_n(x) &= nH_{n-1}(x). && (10) \end{aligned}$$

We define auxiliary functions

$$\tilde{H}_j(x) = \phi(x)H_j(x), \tag{11}$$

which verify the condition

$$\frac{d}{dx}\tilde{H}_n(x) = -\tilde{H}_{n+1}(x). \tag{12}$$

Using this last relation, integration by parts, recursion and the identity

$$\int_{-\infty}^{x} e^{\gamma u}\phi(u)du = e^{\frac{\gamma^2}{2}}\Phi(x-\gamma) \tag{13}$$

we get

$$
\begin{aligned}
&\int_{-\infty}^{x} e^{\gamma u} \tilde{H}_n(u) du \\
&\quad = -e^{\gamma u} \tilde{H}_{n-1}(u) \Big|_{-\infty}^{x} + \gamma \int_{-\infty}^{x} e^{\gamma u} \tilde{H}_{n-1}(u) du \\
&\quad = -e^{\gamma x} \tilde{H}_{n-1}(x) \\
&\quad\quad - \gamma e^{\gamma x} \tilde{H}_{n-2}(x) + \gamma^2 \int_{-\infty}^{x} e^{\gamma u} \tilde{H}_{n-2}(u) du \\
&\quad = \dots \\
&\quad = \gamma^n e^{\frac{\gamma^2}{2}} \Phi(x-\gamma) - e^{\gamma x} \sum_{j=1}^{n} \gamma^{j-1} \tilde{H}_{n-j}(x). \quad (14)
\end{aligned}
$$

## Gram-Charlier expansion

For a density $f$, the Gram-Charlier expansion has the form

$$f(x) = \phi(x)\left(1 + \sum_{j=1}^{\infty} c_j H_j(x)\right) = \tilde{H}_0(x) + \sum_{j=1}^{\infty} c_j \tilde{H}_j(x), \tag{15}$$

where the coefficients $c_j$ are related to the cumulants of $f$. For standardised $f$ (zero mean, unit variance) $c_1 = c_2 = 0$.

Some care is required to ensure the convergence of the formal expansion (15), and the non-negativity of the resulting density $f$. Setting these issues aside, we simply note that (15) provides a convenient and flexible way of constructing densities. Of particular interest are departures from normality manifested in skewness or excess curtosis. Putting

$$\begin{aligned} c_3 &= \frac{\mu_3}{6} && (16) \\ c_4 &= \frac{\mu_4}{24} && (17) \\ c_j &= 0 \text{ otherwise} && (18) \end{aligned}$$

results in a standardised distribution with skewness $\mu_3$ and excess kurtosis $\mu_4$, provided the values as such that $f \geq 0$ everywhere.

**Lemma .1** *Suppose the random variable $X$ has the probabilty density function $f$ of the form (15). Then the following hold*

$$F(x) = \Phi(x) - \sum_{j=1}^{\infty} c_j \tilde{H}_{j-1}(x) \qquad (19)$$

$$1 - F(x) = \Phi(-x) + \sum_{j=1}^{\infty} c_j \tilde{H}_{j-1}(x) \qquad (20)$$

$$M(\gamma) = \mathbf{E}[\mathrm{exp}(\gamma X)] = e^{\frac{\gamma^2}{2}} \left( 1 + \sum_{j=1}^{\infty} c_j \gamma^j \right) \quad (21)$$

$$\int_{-\infty}^{x} e^{\gamma u} f(u) du$$
$$= \Phi(x-\gamma)\mathbf{E}[\exp(\gamma X)] - e^{\gamma x} \sum_{j=1}^{\infty} c_j \sum_{k=1}^{j} \gamma^{k-1} \tilde{H}_{j-k}(x) \tag{22}$$

$$\int_{x}^{\infty} e^{\gamma u} f(u) du$$
$$= \Phi(\gamma-x)\mathbf{E}[\exp(\gamma X)] + e^{\gamma x} \sum_{j=1}^{\infty} c_j \sum_{k=1}^{j} \gamma^{k-1} \tilde{H}_{j-k}(x) \tag{23}$$

*for any real $x$ and $\gamma > 0$.*

All of these follow easily from (14).

We put

$$e^{\mu} = \mathbf{E}[\exp(\gamma X)] = e^{\frac{\gamma^2}{2}} \left(1 + \sum_{j=1}^{\infty} c_j \gamma^j\right) \tag{24}$$

by (21).

## Futures and Options

Suppose we use $\gamma X$ as a model for the log-return on the futures price for some delivery time $T$, where $X$ is a standardised random variable. We shall assume that under the pricing measure the density $f$ of $X$ has the Gram-Charlier expansion as in (15).

By homogeneity we can assume WLOG that the current futures price is 1. We shall thus model the $T$-delivery futures price as

$$F_T = e^{\gamma X - \mu}. \tag{25}$$

The price of a European call on futures, expiring at $T$ and with strike $K$ is given by

$$
\begin{aligned}
&C(K) = e^{-rT}\mathbf{E}[(\exp(\gamma X - \mu) - K)^{+}] \\
&= e^{-rT}\left[\int_{\frac{\log K+\mu}{\gamma}}^{\infty} e^{\gamma x} f(x)dx - K\int_{\frac{\log K+\mu}{\gamma}}^{\infty} f(x)dx\right] \\
&\qquad = e^{-rT}\left[\Phi\left(\gamma - \frac{\log K+\mu}{\gamma}\right)\right. \\
&\qquad + K\sum_{j=1}^{\infty} c_j \sum_{k=1}^{j} \gamma^{k-1}\tilde{H}_{j-k}\left(\frac{\log K+\mu}{\gamma}\right) \\
&\left. -K\left(\Phi\left(-\frac{\log K+\mu}{\gamma}\right) + \sum_{j=1}^{\infty} c_j \tilde{H}_{j-1}\left(\frac{\log K+\mu}{\gamma}\right)\right)\right]
\end{aligned}
\tag{26}
$$

Putting $u = \frac{\log K + \mu}{\gamma}$ we obtain

$$\begin{aligned} C(K) &= e^{-rT}\Bigg[\Phi(\gamma - u) - K\Phi(-u) \\ &+ K\sum_{j=1}^{\infty} c_j\Bigg(-\tilde{H}_{j-1}(u) + \sum_{k=1}^{j} \gamma^{k-1}\tilde{H}_{j-k}(u)\Bigg)\Bigg] \\ &= e^{-rT}\Bigg[\Phi(\gamma - u) - K\Phi(-u) \\ &+ K\sum_{j=1}^{\infty} c_j \sum_{k=2}^{j} \gamma^{k-1}\tilde{H}_{j-k}(u)\Bigg] \\ &= e^{-rT}\Bigg[\Phi(\gamma - u) - K\Phi(-u) \\ &+ K\sum_{k=2}^{\infty} \gamma^{k-1} \sum_{j=k}^{\infty} c_j\tilde{H}_{j-k}(u)\Bigg] \quad (27) \end{aligned}$$

From (24) we have

$$\mu = \frac{\gamma^2}{2}\log\Bigg(1 + \sum_{j=1}^{\infty} c_j\gamma^j\Bigg) = \frac{\gamma^2}{2}\log(1+\chi), \quad (28)$$

and hence

$$
\begin{aligned}
u &= \frac{\log K + \mu}{\gamma} = \frac{\log K(1+\chi)}{\gamma} + \frac{\gamma}{2}, && (29)\\
\gamma - u &= \frac{-\log K(1+\chi)}{\gamma} + \frac{\gamma}{2}. && (30)
\end{aligned}
$$

Comparing these with (2)-(3) we observe

$$
\begin{aligned}
u &= -d_2(K(1+\chi), \gamma T^{-1/2}) && (31)\\
\gamma - u &= d_1(K(1+\chi), \gamma T^{-1/2}). && (32)
\end{aligned}
$$

Rewriting (27) yields

$$
\begin{aligned}
&C(K) = C_{BS}(K(1+\chi), \gamma T^{-1/2})\\
&\quad + e^{-rT}\left[K\phi(-d_2)\sum_{k=2}^{\infty}\gamma^{k-1}\sum_{j=k}^{\infty}c_j H_{j-k}(-d_2)\right],
\end{aligned}
\tag{33}
$$

where we drop the arguments of $d_2$ for legibility.

Since $\phi$ is even, and using the identity $K\phi(d_2) = F\phi(d_1)$, we rewrite (34) to obtain

$$C(K) = C_{BS}(K(1+\chi), \gamma T^{-1/2}) + e^{-rT}\phi(d_1)\frac{1}{1+\chi}\sum_{k=2}^{\infty}\gamma^{k-1}\sum_{j=k}^{\infty}c_j H_{j-k}(-d_2). \quad (34)$$

## Volatility skew and pricing density

We now once again turn our attention to the implied volatility. First order Taylor's expansion in $\sigma$, applied to (1) yields

$$\begin{aligned} C_{BS}(K, \sigma_{ATM} + \alpha(m(K))) &= C_{BS}(K, \sigma_{ATM}) \\ &+ \frac{\partial}{\partial \sigma} C_{BS}(K, \sigma)\Big|_{\sigma = \sigma_{ATM}} \alpha(m(K)) + O(\alpha(m(K))^2) \\ &= C_{BS}(K, \sigma_{ATM}) \\ &+ e^{-rT} \phi(d_1(K, \sigma_{ATM})) T^{1/2} \alpha(m(K)) \\ &+ O(\alpha(m(K))^2). \end{aligned} \tag{35}$$

For clarity we rewrite (35) and (34).

$$C_{BS}(K) \approx C_{BS}(K, \sigma_{ATM}) \\ + e^{-rT}\phi[d_1(K, \sigma_{ATM})]T^{1/2}\alpha[m(K)] \quad (36)$$

$$C(K) = C_{BS}(K(1+\chi), \gamma T^{-1/2}) \\ + e^{-rT}\phi[d_1(K(1+\chi), \gamma T^{-1/2})]T^{1/2}\alpha_0[m(K(1+\chi))] \quad (37)$$

with

$$\alpha_0[m(K(1+\chi))] \\ = \frac{1}{T^{1/2}(1+\chi)} \sum_{k=2}^{\infty} \gamma^{k-1} \sum_{j=k}^{\infty} c_j H_{j-k}(-d_2 m(K(1+\chi))) \quad (3$$

where $K$ only appears in $d_2$, which is a linear fuction of $m(K)$.

Comparing (36) and (37) we observe that the theoretically correct pricing formula (37) is equivalent (up to first order) to a skew-corrected Black-Scholes formula (36) with

- a slightly modified strike, from $K$ to $K(1+\chi)$,

- $\gamma$ playing the role of $\sigma_{ATM}T^{1/2}$, and

- with the skew function given by (38).

## Example

With the density $f$ given by (16)-(18), the skew $\alpha_0$ becomes the second degree polynomial in $d_2$:

$$\frac{\sigma_{ATM}T^{1/2}}{T^{1/2}\left(1+c_3\sigma_{ATM}^3T^{3/2}+c_4\sigma_{ATM}^4T^2\right)} \left(c_3\sigma_{ATM}T^{1/2}+c_4(\sigma_{ATM}^2T-1) - (c_3+c_4\sigma_{ATM}T^{1/2})d_2 + c_4d_2^2\right) \quad (39)$$

This is a little better than a “wrong number in wrong formula,” for a few reasons.

- Our results justify the approach of modelling the skew as a function of moneyness which does not change with $F$ (although strike adjustments may be required).

- ATM implied volatiltiy is shown to be directly connected to the variance of the pricing density.

- The form of the skew function can be related to the cumulants of the pricing density.

- The practice of using polynomials for skew can be viewed as equivalent to truncating the Gram-Charlier expansion of the pricing density.