1. Pricing European options based on the fuzzy pattern of Black-Scholes formula [Wu 2004]

2. Using fuzzy sets theory and Black-Scholes formula to generate pricing boundaries of European options [Wu 2007]

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Outline

- Introduction
- Method and Result [Wu 2004]
- Method and Result [Wu 2007]
- Remarks

Uncertainties

In real world, the data sometimes cannot be recorded or collected precisely. For instance,

- the water level of a river cannot be measured in an exact way because of the fluctuation.
- the price of a commodity fluctuates from time to time owing to the market effects.
- the interest rate may have different values in different commercial banks and financial institutions.

Modeling uncertaities

Fuzzy sets theory proposed by Zadeh (1965) provides an appropriate tool in modeling uncertainties. It is appropriate to say

- the water level is around 30 m.
- the price of a stock is $\$32 - \$34$.
- the interest rates may be around 5%.

All of those statements can be characterized as fuzzy sets.

Fuzzy sets

- The fuzzy set $\tilde{A}$ is defined by its membership function $\mu_{\tilde{A}} : X \to [0, 1]$ which is viewed as an extension of the characteristic function.

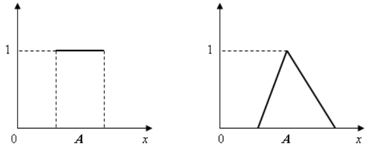
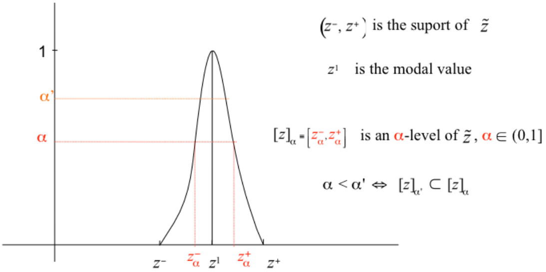


Figure: Characteristic function (left) vs Membership function (right).

Definitions

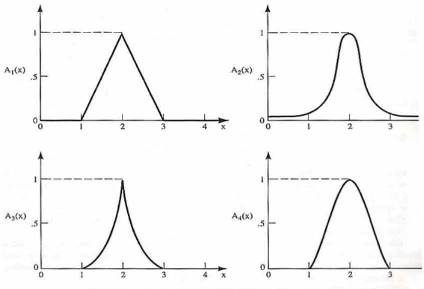
- We denote the α-level set $\tilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\}$



Fuzzy number

The fuzzy set $\tilde{A}$ is called a fuzzy number if

- $\tilde{A}$ is a normal set, i.e., $\exists x$ s.t. $\mu_{\tilde{A}}(x) = 1$.
- $\tilde{A}$ is a convex set, i.e.
  $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y))$.
- $\tilde{A}_\alpha$ is closed for each α and $\tilde{A}_0$ is bounded.



Examples of membership functions that may be used in different contexts for

Models [Wu 2004] and [Wu 2007]

A European call option on a stock S with expiry date T and strike price K.

- The Black-Scholes formula for the option price at t is:

$$C_t = f(S_t, T - t, K, r, \sigma)$$

where

$$f(s, t, K, r, \sigma) = sN(d_1) - Ke^{-rt}N(d_2),$$

$$d_1 = \frac{ln(s/K)+(r+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \text{ and } d_2 = d_1 - \sigma\sqrt{t}.$$

- Under the considerations of fuzzy interest rate $\tilde{r}$, fuzzy volatility $\tilde{\sigma}$ and fuzzy stock price $\tilde{S}$, the fuzzy price $\tilde{C}_t$ is obtained in Wu [2004] and Wu [2007] (see below).

Model Continued $\cdots$

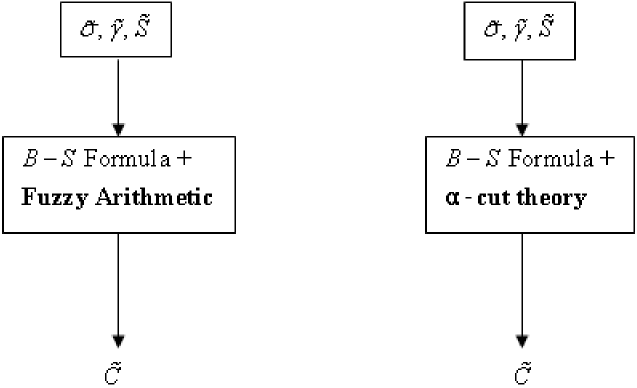


Figure: Wu 2004 (left) and Wu 2007 (right).

Method [Wu 2004]

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then $\tilde{a} \oplus \tilde{b}$, $\tilde{a} \ominus \tilde{b}$, $\tilde{a} \otimes \tilde{b}$ are fuzzy numbers and their α-level sets are

- $(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U]$
- $(\tilde{a} \ominus \tilde{b})_\alpha = [\tilde{a}_\alpha^L - \tilde{b}_\alpha^U, \tilde{a}_\alpha^U - \tilde{b}_\alpha^L]$
- $(\tilde{a} \otimes \tilde{b})_\alpha = [\min\{\tilde{a}_\alpha^L \tilde{b}_\alpha^L, \tilde{a}_\alpha^L \tilde{b}_\alpha^U, \tilde{a}_\alpha^U \tilde{b}_\alpha^L, \tilde{a}_\alpha^U \tilde{b}_\alpha^U\}, \max\{\tilde{a}_\alpha^L \tilde{b}_\alpha^L, \tilde{a}_\alpha^L \tilde{b}_\alpha^U, \tilde{a}_\alpha^U \tilde{b}_\alpha^L, \tilde{a}_\alpha^U \tilde{b}_\alpha^U\}]$
- The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\mu_{\tilde{a} \odot \tilde{b}}(z) = \sup_{(x,y): x \circ y = z} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}$$

where $\odot = \oplus, \ominus, \otimes$ or $\oslash$ correspond to $\circ = +, -, \times$ or $/$.

Results [Wu 2004]

Applying to the B-S formula,

- we have

$$\tilde{f}(\tilde{s}, t, K, \tilde{r}, \tilde{\sigma}) = \tilde{s} \otimes \tilde{N}(\tilde{d}_1) \ominus Ke^{-\tilde{r}t} \otimes \tilde{N}(\tilde{d}_2)$$

where $\tilde{d}_1 = \frac{\ln(\tilde{s}/K) \oplus (\tilde{r} \oplus \frac{\tilde{\sigma} \otimes \tilde{\sigma}}{2})t}{\tilde{\sigma}\sqrt{t}}$ and $\tilde{d}_2 = \tilde{d}_1 \ominus \tilde{\sigma}\sqrt{t}$.

- Then the fuzzy price

$$\tilde{C}_t = \tilde{f}(\tilde{S}_t, T - t, K, \tilde{r}, \tilde{\sigma}).$$

Results [Wu 2004] continued $\cdots$

- Given any price c, the membership value is:

$$\mu_{\tilde{C}_t}(c) = \sup_{(r,\sigma,s):c=f(s,T-t,K,r,\sigma)} \min\{\mu_{\tilde{\mathrm{r}}}(\mathrm{r}), \mu_{\tilde{\sigma}}(\sigma), \mu_{\tilde{\mathrm{S}}}(\mathrm{s})\}$$

- Given any membership value α, the α-level set is:

$$(\tilde{C}_t)_\alpha = [(\tilde{C}_t)^L_\alpha, (\tilde{C}_t)^U_\alpha]$$

where

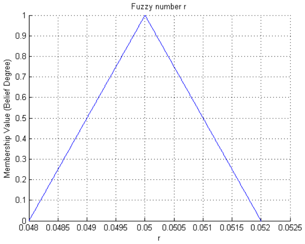
$$(\tilde{C}_t)^L_\alpha = (\tilde{S}_t)^L_\alpha N((\tilde{d}_1)^L) - Ke^{-\tilde{r}^L_\alpha(T-t)} N((\tilde{d}_2)^U_\alpha)$$

and

$$\tilde{C}_t)^U_\alpha = (\tilde{S}_t)^U_\alpha N((\tilde{d}_1)^U) - Ke^{-\tilde{r}^U_\alpha(T-t)} N((\tilde{d}_2)^L_\alpha).$$

Numerical Results [Wu 2004]

European call option, $T = 0.25$, $K = \$30$, $\tilde{r} = (0.048, 0.05, 0.052)$ and $\tilde{\sigma} = (0.08, 0.1, 0.12)$, $\tilde{S}_0 = (32, 33, 34)$.



Numerical Results [Wu 2004] continued $\cdots$

α	$\tilde{C}_{0\alpha}$
0.99	[3.3453,3.4174]
0.98	[3.3092,3.4534]
0.97	[3.2732,3.4895]
0.96	[3.2372,3.5256]
0.95	[3.2011,3.5617]
$\cdots$	$\cdots$
0.90	[3.0205,3.7427]

For $\alpha = 0.95$, it means that the call option price will lie in the closed interval $[3.2011, 3.5617]$ with belief degree 0.95.

Method [Wu 2007]

- Let $f(x_1, x_2, \cdots, x_n)$ be a continuous real-valued function defined on $\mathbb{R}^n$ and $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ be n fuzzy numbers.
- Let $\tilde{f} : \mathcal{F}^n \rightarrow \mathcal{F}$ be a fuzzy-valued function induced by $f(x_1, x_2, \cdots, x_n)$ via the extension principle defined below:

$$\mu_{\tilde{f}(\tilde{A}_1,\cdots,\tilde{A}_n)}(r) = \sup_{\{(x_1,\cdots,x_n):r=f(x_1,\cdots,x_n)\}} \min\{\mu_{\tilde{\mathrm{A}}_1}(\mathrm{x}_1), \cdots, \mu_{\tilde{\mathrm{A}}_\mathrm{n}}(\mathrm{x}_\mathrm{n})$$

- Suppose that each $\{(x_1, x_2, \cdots, x_n) : r = f(x_1, x_2, \cdots, x_n)\}$ is a compact subset of $\mathbb{R}^n$ for r in the range of f.
- Then $\tilde{f}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ is a fuzzy number and its α-level set is

$$\tilde{f}(\tilde{a}_1, \cdots, \tilde{a}_n)_\alpha = \{f(x_1, \cdots, x_n) : x_1 \in (\tilde{a}_1)_\alpha, \cdots, x_n \in (\tilde{a}_n)_\alpha\}.$$

Results [Wu 2007]

Applying to the B-S formula

- Given any price c, its membership value is:

$$\mu_{\tilde{C}_t}(c) = \sup_{\{(s,r,\sigma):c=f(s,T-t,K,r,\sigma)\}} \min\{\mu_{\tilde{\mathrm{S}}_\mathrm{t}}(\mathrm{s}), \mu_{\tilde{\mathrm{r}}}(\mathrm{r}), \mu_{\tilde{\sigma}}(\sigma)\}.$$

- Given any membership value α, its α-level set is:

$$(\tilde{C}_t)_\alpha = \{f(s, T-t, K, r, \sigma) : s \in (\tilde{S}_t)_\alpha, r \in \tilde{r}_\alpha, \sigma \in \tilde{\sigma}_\alpha\}$$

where

$$(\tilde{C}_t)^L_\alpha = \min_{s\in\tilde{S}_\alpha, r\in\tilde{r}_\alpha, \sigma\in\tilde{\sigma}_\alpha} f(s, T-t, K, r, \sigma)$$

and

$$(\tilde{C}_t)^U_\alpha = \max_{s\in\tilde{S}_\alpha, r\in\tilde{r}_\alpha, \sigma\in\tilde{\sigma}_\alpha} f(s, T-t, K, r, \sigma).$$

Numerical Results [Wu 2007]

European call option, $T = 0.25$, $K = \$30$, $\tilde{r} = (0.048, 0.05, 0.052)$ and $\tilde{\sigma} = (0.08, 0.1, 0.12)$, $\tilde{S}_0 = (32, 33, 34)$.

α	$\tilde{C}_{0\alpha}$
0.99	[3.3712,3.3914]
0.98	[3.3611,3.4016]
0.97	[3.3509,3.4117]
0.96	[3.3408,3.4218]
0.95	[3.3307,3.4319]
...	...
0.90	[3.2801,3.4825]

For $\alpha = 0.95$, it means that the price will lie in the interval $[3.3307, 3.4319]$ with belief degree 0.95.

Remarks

- Method [Wu 2007] is more reliable than that of [Wu 2004].
- With the fuzzy price, the financial analyst can pick any price with an acceptable belief degree for later use.
- The fuzzy number $\tilde{r}$ may be developed through the real data, which may better fit the real world.
- $\tilde{S}$ and $\tilde{\sigma}$ may be modeled with a fuzzy random number which can be obtained by calibrating the model.
- The method developed in [Wu 2007] can be extended to a fuzzy PDE model.

Remarks continued $\cdots$

- Kejiang and I have proposed the fuzzy PDE method for modeling contaminant transport process. For instance,

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} - \lambda C$$
$$C(0,t) = C_0, \quad \frac{\partial C}{\partial x}|_{\to\infty} = 0 \quad t > 0, \quad C = 0 \quad t = 0. \tag{1}$$

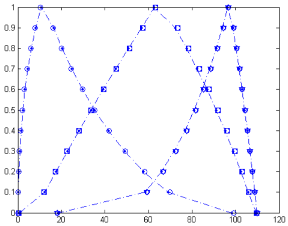


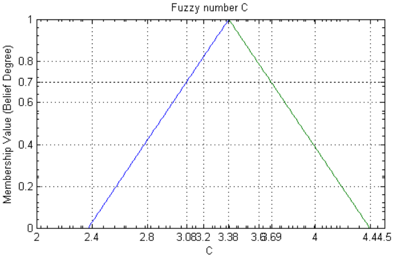
Figure: C at $x = 10$, $t = 5$(left) , $t = 10$ (middle) and $t = 20$ (right).

Remarks

- Our method has been extended to solve the B-S PDE

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial t^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$
$$V(0,t) = 0, \quad V(S,t) \approx S - Ke^{r(T-t)} \quad \text{as} \quad S \to \infty \qquad (2)$$
$$V(S,T) = \max(S-K,0).$$

- The same case as in Wu [2007] is studied.



End of the talk

We are going to explore more !

- Building a fuzzy (stochastic) model
- Developing some fuzzy numbers from real data
- Calibrating the fuzzy model
- Fast algorithm for solving the fuzzy model
- Risk management based on data obtained from the fuzzy model

We invite you to join us! Please email to hnli@ucalgary.ca.

Thank you very much!