Stochastic Modelling of Electricity and Related Markets: Chapter 3

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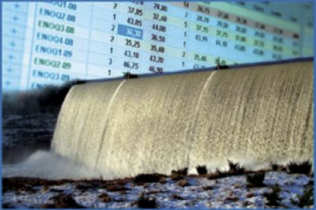
STOCHASTIC MODELLING OF ELECTRICITY AND RELATED MARKETS



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World Scientific

Stochastic models for energy spot price dynamics

The Schwartz model

$$S(t) = S(0)\exp X(t), \quad \text{with} \quad dX(t) = \kappa\big(\alpha - X(t)\big)dt + \sigma dW(t).$$

The starting point for the models in this chapter (and indeed for much of commodity spot price modelling) is the Schwartz one-factor model from his 1997 paper
The stochastic behavior of commodity prices: implications for valuation and hedging
(Journal of Finance, Vol. 52(3), 923–973).

Spot price modelling with OU processes

$I(t)$ is an II process with a Lévy-Kintchine representation

$$\psi(t,s;\theta) = i\theta\big(\gamma(s)-\gamma(t)\big) - \frac{1}{2}\theta^2\big(C(s)-C(t)\big) + \int_t^s \int_{\mathbb{R}} \left\{ \mathrm{e}^{\mathrm{i}z\theta} - 1 - \mathrm{i}z\theta \mathbf{1}_{|z|<1} \right\} l(dz,du),$$

where γ is of finite variation.

An RCLL process $X(s)$ ($t \leq s \leq T$) is an OU process if it is the unique strong solution to

$$dX(s) = \big(\mu(s) - \alpha(s)X(s)\big)ds + \sigma(s)dI(s), \qquad X(t) = x.$$

The unique solution can be written

$$X(s) = x\,\mathrm{e}^{-\int_t^s \alpha(v)dv} + \int_t^s \mu(u)\mathrm{e}^{-\int_u^s \alpha(v)dv}du + \int_t^s \sigma(u)\mathrm{e}^{-\int_u^s \alpha(v)dv}dI(u).$$

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Spot price modelling with OU processes

The characteristic function of an OU process

$$\mathbb{E}\left[e^{i\theta X(s)}|X(t)=x\right]=e^{\chi(t,s;\theta)},$$

with

$$\chi(t,s;\theta)=i\theta\left(x\,e^{-\int_t^s\alpha(v)dv}+\int_t^s\mu(u)e^{-\int_u^s\alpha(v)dv}du\right)\\+\psi\left(t,s;\theta\sigma(\cdot)e^{-\int_\cdot^s\alpha(v)dv}\right),$$

where $\psi\big(t,s;g(\cdot)\big)$ is defined by

$$\psi\big(t,s;g(\cdot)\big)=i\int_t^s g(u)d\gamma(u)-\frac{1}{2}\int_t^s g^2(u)dC(u)\\+\int_t^s\int_{\mathbb{R}}\left\{e^{izg(u)}-1-izg(u)\mathbf{1}_{|z|<1}\right\}l(dz,du).$$

Spot price modelling with OU processes

The expected value of an OU process

The above result can be used to show that, if $\int_t^s \int_{|z|\geq 1} |z|\, l(dz, du) < \infty$,

$$\mathbb{E}\big[X(s)|X(t) = x\big] = x\,\mathrm{e}^{-\int_t^s \alpha(v)dv} + \int_t^s \mu(u)\mathrm{e}^{-\int_u^s \alpha(v)dv} du + \int_t^s \sigma(u)\mathrm{e}^{-\int_u^s \alpha(v)dv} d\gamma u + \int_t^s \int_{|z|\geq 1} z\sigma(u)\mathrm{e}^{-\int_u^s \alpha(v)dv} l(dz, du).$$

Brownian motion

If $I(t) = B(t)$, then $X(s)$ (conditioned on $X(t) = x$) is normal with mean and variance

$$x\,\mathrm{e}^{-\int_t^s \alpha(v)dv} + \int_t^s \mu(u)\mathrm{e}^{-\int_u^s \alpha(v)dv} du \quad \text{and} \quad \int_t^s \sigma^2(u)\mathrm{e}^{-2\int_u^s \alpha(v)dv} du.$$

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Spot price modelling with OU processes

The stationary distribution of an OU process

Suppose that all coefficients are constant. In the Brownian motion case, we find that

$$\lim_{s\to\infty} X(s) = X_\infty,$$

where

$$X_\infty \sim N\left(\frac{\mu}{\alpha}, \frac{\sigma^2}{2\alpha}\right).$$

If $I(t) = L(t)$ (a Lévy process), and $\int_{|z|\geq 2} \ln|z|\,\tilde{l}(dz) < \infty$, then the cumulant function of X_∞ is

$$\mathrm{i}\theta\frac{\mu}{\alpha} + \int_0^\infty \psi(\theta \mathrm{e}^{-\alpha s})ds,$$

where $\phi(\theta)$ is the cumulant function of $L(1)$.

Geometric models

Introduce n independent pure jump semimartingale II processes $I_j(t)$, given by

$$I_j(t) = \gamma_j(t) + \int_0^t \int_{|z|<1} z\,\tilde{N}_j(dz,du) + \int_0^t \int_{|z|\geq 1} z\,N_j(dz,du),$$

and p independent Brownian motions $B_j(t)$.
Define $S(t)$ by

$$\ln S(t) = \ln \Lambda(t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t),$$

where

$$dY_j(t) = \big(\delta_j(t) - \beta_j(t)Y_j(t)\big)dt + \eta_j(t)dI_j(t),$$

and

$$dX_i(t) = \big(\mu_i(t) - \alpha_i(t)X_i(t)\big)dt + \sum_{k=1}^{p} \sigma_{ik}(t)dB_k(t),$$

with $\Lambda(t)$ modelling the seasonal price level.

Geometric models

The dynamics of $S(t)$ are given by

$$\frac{dS(t)}{S(t-)} = \Bigg\{ \frac{\Lambda'(t)}{\Lambda(t)} + \frac{1}{2}\sum_{i,j,k} \sigma_{ij}(t)\sigma_{jk}(t) + \sum_i \big(\mu_i(t) - \alpha_i(t)X_i(t)\big) + \sum_j \big(\delta_j(t) - \beta_j(t)Y_j(t)\big) \Bigg\} dt + \sum_j \int_{|z|<1} \left\{ e^{\eta_j(t)z} - 1 - \eta_j(t)z \right\} l_j(dz,dt) + \sum_{i,k} \sigma_{ik} dB_k(t) + \sum_j \int_{|z|<1} \left\{ e^{\eta_j(t)z} - 1 \right\} \tilde{N}_j(dz,dt) + \sum_j \int_{|z|\geq 1} \left\{ e^{\eta_j(t)z} - 1 \right\} N_j(dz,dt).$$

Integrability conditions apply if we are to use this for option pricing.

In the one-factor Schwartz case this reduces to

$$\frac{dS(t)}{S(t)} = \left\{ \frac{\Lambda'(t)}{\Lambda(t)} + \alpha(t)\ln\Lambda(t) + \frac{1}{2}\sigma^2(t) + \big(\mu(t) - \alpha(t)\ln S(t)\big) \right\} dt + \sigma(t)dB(t).$$

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$$\left. + \sum_i \big(\mu_i(t) - \alpha_i(t)X_i(t)\big) + \sum_j \big(\delta_j(t) - \beta_j(t)Y_j(t)\big) \right\} dt$$
$$+ \sum_j \int_{|z|<1} \left\{ \mathrm{e}^{\eta_j(t)z} - 1 - \eta_j(t)z \right\} l_j(dz,dt) + \sum_{i,k} \sigma_{ik} dB_k(t)$$
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Geometric models

Some more special cases

$$dX(t) = -\alpha(t)X(t)dt + \sigma(t)dB(t)$$
$$dY(t) = -\alpha(t)Y(t)dt + dI(t).$$
$$d\ln S(t) = d\ln\Lambda(t) - \alpha(t)\big(\ln S(t) - \ln\Lambda(t)\big)dt + \sigma(t)dB(t) + dI(t).$$

- Benth and Šaltytė Benth (2004)
- Pure jump NIG

Lucia and Schwartz (2002)

$$dX_1(t) = -\alpha_1 X_1(t)dt + \sigma_1 dB_1(t)$$
$$dX_2(t) = \mu_2 dt + \sigma_2\left(\rho dB_1(t) + \sqrt{1-\rho^2}dB_2(t)\right)$$

Villaplana (2002) replaces μ_2 by $\big(\mu_2 - \alpha_2 X_2(t)\big)$, and adds

$$dY(t) = -\alpha_1 Y(t)dt + dI(t),$$

with $I(t)$ a time-inhomogenous compound Poisson process.

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- Eberlein Stahl (2003)
- No mean-reversion (in $Y(t)$), $I(t)$ hyperbolic Lévy

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- Cartea and Figueroa (2005)
- $I(t)$ compound Poisson

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- Geman and Roncoroni (2006)
- $dI(t) = h(S(t))dJ(t)$, $J(t)$ time-inhomogenous compound Poisson.

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Arithmetic models

Here

$$S(t) = \Lambda(t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_j(t),$$

where $X_i(t)$ and $Y_j(t)$ are as before. Again, integrability conditions will apply.

Negative values

If $Y_j(t) = 0, j = 1, \ldots, n$, then $S(t)$ is a Gaussian OU process (mixture) and can become negative. In fact,

$$\mathbb{P}[S(t) < 0] = \Phi\left(-\frac{m(t)}{\Sigma(t)}\right),$$

where

$$m(t) = \Lambda(t) + \sum_i X_i(0)\mathrm{e}^{-\int_0^t \alpha_i(s)ds}$$

$$\Sigma^2(t) = \sum_k \int_0^t \sigma_{ik}^2(s)\mathrm{e}^{-2\int_s^t \alpha_i(u)du}ds.$$

Arithmetic models

The nonnegative model of Benth, Kallsen and Meyer-Brandis (2007)

- $m = 0$.
- The pure jump processes $I_j(t)$ are *increasing*.
- The mean-reverting levels $\delta_j(t) = 0$.
- $\Lambda(t)$ is now interpreted as a *seasonal floor* for $S(t)$.

The mean level of spot prices is given by

$$\Lambda_m(t) = \Lambda(t) + Y_1(0)\mathrm{e}^{-\int_0^t \beta_1(v)dv} + \sum_j \int_0^t \eta_j(u)\mathrm{e}^{-\int_u^t \beta_1(v)dv} d\tilde{\gamma}_j(u)$$

$$+ \sum_j \int_0^t \int_0^\infty z\eta_j(u)\mathrm{e}^{-\int_u^t \beta_1(v)dv} l_j(dz, du),$$

where $\tilde{\gamma}_j(t) = \gamma_j(t) + \int_0^t \int_0^1 z l_j(dz, du)$.

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where $\tilde{\gamma}_j(t) = \gamma_j(t) + \int_0^t \int_0^1 z l_j(dz, du)$.

The autocorrelation function of multi-factor OU processes

Let $Z(t)$ be a deseasonal additive OU process:

$$Z(t) = \sum_i X_i(t) + \sum_j Y_j(t) \quad \text{with constant coefficients.}$$

The **autocorrelation function** at time t with lag τ is given by

$$\rho(t,\tau) = \mathrm{Corr}[Z(t), Z(t+\tau)].$$

$$\rho(t,\tau) = \sum_i \hat{\omega}_i(t,\tau)\mathrm{e}^{-\alpha_i\tau} + \sum_j \tilde{\omega}_j(t,\tau)\mathrm{e}^{-\beta_j\tau},$$

where

$$\hat{\omega}_i(t,\tau) = \frac{\sum_{i'} \frac{\sum_k \sigma_{ik}\sigma_{i'k}}{\alpha_i+\alpha_{i'}}\left(1-\mathrm{e}^{(\alpha_i+\alpha_{i'})t}\right)}{\sqrt{\mathrm{Var}[Z(t+\tau)]\mathrm{Var}[Z(t)]}}$$

$$\tilde{\omega}_j(t,\tau) = \frac{\mathrm{Var}[Y_j(t)]}{\sqrt{\mathrm{Var}[Z(t+\tau)]\mathrm{Var}[Z(t)]}}.$$

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Simulation of stationary OU processes: a case study of with the arithmetic spot model

$$S(t) = \Lambda(t) + Y_1(t) + Y_2(t).$$

$$\Lambda(t) = 100 + 0.025t + 30 \sin \frac{2\pi t}{365}.$$

$$Y_1(t+\Delta) = \mathrm{e}^{-\beta\Delta}\left(Y_1(t) + Z(t)\right),$$

where

$$Z(t) = \int_0^{\Delta} \mathrm{e}^{\beta_1 u} dL(u) = \mu_J \sum_{1}^{N(1)} \ln(c_i^{-1}) \mathrm{e}^{\beta_1 \Delta u_i},$$

with u_i independent samples from $U([0,1])$, c_i arrival times of a Poisson process with intensity $\nu\beta_1\Delta$, $N(1)$ the number of jumps up to time 1, with $\Delta = 1$, $\nu = 8.06$, $\beta_1 = 0.085$, $\mu_J = 7.7$.

$Y_2(t)$ is an inhomogeneous compound Poisson process, with exponential jump sizes with mean 180, and intensity $\lambda(t) = \frac{0.14}{\left|\sin\left(\frac{\pi(t-90)}{365}\right)\right| + 1} - 1.$

Simulation of stationary OU processes: a case study of with the arithmetic spot model

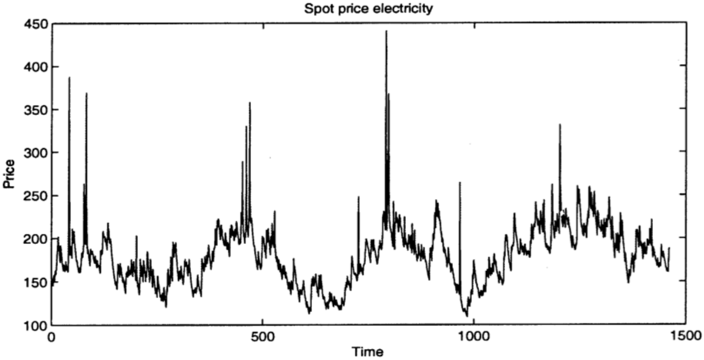

Fig. 3.1 Four years of daily spot prices simulated from the arithmetic model with seasonal spikes defined in (3.35).

Simulation of stationary OU processes: a case study of with the arithmetic spot model

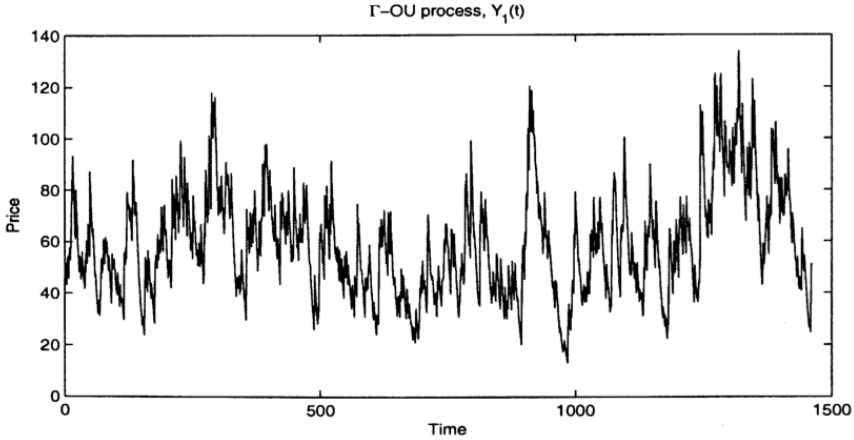

Fig. 3.2 The Gamma OU process $Y_1(t)$.

Simulation of stationary OU processes: a case study of with the arithmetic spot model

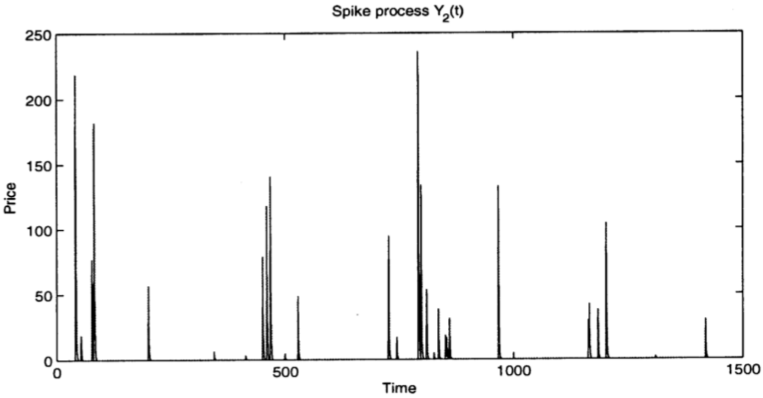

Fig. 3.3 The OU process $Y_2(t)$ with seasonal intensity for jumps given in (3.36).